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Constructing the large mixing angle MNS matrix in see-saw models with right-handed neutrino dominance

Stephen F. King

Department of Physics and Astronomy, University of Southampton
Southampton, SO17 1BJ, UK
E-mail: sfk@hep.phys.soton.ac.uk

Abstract: Recent SNO results strongly favour the large mixing angle (LMA) MSW solar solution. We argue that there are only two technically natural low energy neutrino mass matrix structures consistent with the LMA MSW solution, corresponding to either a hierarchy or an inverted hierarchy with pseudo-Dirac neutrinos. We construct the MNS matrix to leading order in the small angle θ_{13} including the neutrino *and* charged lepton mixing angles and phases, the latter playing a crucial rôle for allowing the inverted hierarchy case to be consistent with the LMA MSW solution. We then consider the see-saw mechanism with right-handed neutrino dominance and show how the successful neutrino mass matrix structures may be constructed with no tuning and with small radiative corrections, leading to a full, partial or inverted neutrino mass hierarchy. In each case we derive approximate analytic relations between the input see-saw parameters and the resulting neutrino masses, mixing angles and phases, which will provide a useful guide for unified model building. For the hierarchical cases the LMA MSW solution gives a soft lower bound $|U_{e3}| \gtrsim 0.1$, just below the current CHOOZ limit. Both hierarchical and inverted hierarchical cases predict small $\beta\beta_{0\nu}$ with $|m_{ee}| \sim 0.007\,\mathrm{eV}$ within the sensitivity of future proposals such as GENIUS. Successful leptogenesis is possible if the dominant right-handed neutrino is the heaviest one, but the leptogenesis phase is unrelated to the MNS phases.

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1. Introduction

Recent SNO results on the neutral current (NC) flux [1] and the day-night effects [2], when combined with other solar neutrino data especially that of Super-Kamiokande [3] strongly favour the large mixing angle (LMA) MSW solar solution [4] with three active light neutrino states, and $\theta_{12} \approx \pi/6$, $\Delta m^2_{21} \approx 5 \times 10^{-5}\,\mathrm{eV}^2$ [5, 6, 7]. The atmospheric neutrino data is consistent with maximal $\nu_\mu - \nu_\tau$ neutrino mixing [8] with $|\Delta m^2_{32}| \approx 2.5 \times 10^{-3}\,\mathrm{eV}^2$ and the sign of Δm^2_{32} undetermined. The CHOOZ experiment limits $\theta_{13} \lesssim 0.2$ over the favoured atmospheric range [9]. The combined neutrino data is well described by an MNS matrix [10] with $\theta_{23} \approx \pi/4$, $\theta_{12} \approx \pi/6$, $\theta_{13} \lesssim 0.2$, which we refer to as the LMA MNS matrix. It is clear that neutrino oscillations, which only depend on $\Delta m^2_{ij} \equiv m^2_i - m^2_j$, gives no information about the absolute value of the neutrino mass eigenvalues m_i. Recent results from the 2df galaxy redshift survey indicate that $\sum m_i < 1.8\,\mathrm{eV}(95\% C.L.)$ under certain mild assumptions [11]. Combined with the solar and atmospheric oscillation data this brackets the heaviest neutrino mass to be in the approximate range 0.04-0.6 eV. The fact that the mass of the heaviest neutrino is known to within an order of magnitude represents remarkable progress in neutrino physics over recent years.

The basic possible patterns of neutrino mass consistent with this data are: (i) hierarchy (full $m_1 \ll m_2 \ll m_3$, or partial $m_1 \lesssim m_2 \ll m_3$), (ii) inverted hierarchy ($m_1 \approx m_2 \gg m_3$, or the pseudo-Dirac form $-m_1 \approx m_2 \gg m_3$), or (iii) degenerate $m_1^2 \approx m_2^2 \approx m_3^2$ [12]. Although oscillation data does not distinguish between these possibilities, the theoretical requirement that the neutrino spectrum is generated in a technically natural way, does provide an additional guiding principle. For example it is clear that a degenerate mass scale $m^2 \gg |\Delta m^2_{ij}|$ implies very small fractional neutrino mass splittings. For example, if $m_3 = 0.500\,\mathrm{eV}$ then atmospheric oscillations require neutrino masses $m_2 = 0.497\,\mathrm{eV}$. The problem is one of tuning, both to set up the small mass splitting at the high energy scale, and to preserve it in the presence of radiative corrections [13]. In general such a neutrino spectrum is not technically natural, since small perturbations in the high energy input parameters will violate the low energy degeneracy. One can envisage a technically natural mechanism which would lead to a degenerate pair of neutrinos with opposite sign masses $m_1 \approx -m_2$, but to achieve the full three-fold degeneracy is much more difficult [14, 15]. A similar objection can be raised against the inverted hierarchical spectrum in which the almost degenerate neutrinos have the same sign masses $m_1 \approx m_2$ where for $m_1 = 0.0500\,\mathrm{eV}$ we require $m_2 = 0.0497\,\mathrm{eV}$ for the LMA MSW solution. In these kinds of inverted hierarchy models there is no natural mechanism that can set and preserve such mass splittings, however as we shall see, in the inverted hierarchy model with a pseudo-Dirac neutrino pair corresponding to opposite sign masses $m_1 \approx -m_2$, a natural mechanism is possible.

Assuming the LMA MNS matrix plus naturalness implies that there are then just two leading order forms for the light physical effective Majorana neutrino mass matrix m^ν_{LL} (where LL means that it couples left-handed neutrinos to left-handed neutrinos) corresponding to either a hierarchical spectrum with $m \approx m_3 \gg m_1, m_2$ arising from:

$$m^\nu_{LL} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \frac{m}{2} \tag{1.1}$$

or an inverted hierarchical spectrum with a pseudo-Dirac pair of neutrinos with $m \approx m_2 \approx -m_1 \gg m_3$ arising from:

$$m^\nu_{LL} \approx \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \frac{m}{\sqrt{2}}, \tag{1.2}$$

where in both cases $m \approx \sqrt{|\Delta m^2_{32}|} \approx 5 \times 10^{-2}\,\mathrm{eV}$. It is remarkable that only by assuming the LMA MNS matrix plus naturalness arguments we are led to only two possible leading order structures for the neutrino mass matrix, both of which have a zero in the 11 element corresponding to a small $\beta\beta_{0\nu}$ rate corresponding to $|m_{ee}| = 0$ at leading order.

For the hierarchical case the experimental requirement $|\Delta m^2_{32}| \gg |\Delta m^2_{21}|$ implies that $m_3^2 \gg m_2^2$ which again looks technically unnatural since we would expect two roughly equal eigenvalues once the lower block is diagonalised. For the LMA MSW solution we only require a mild hierarchy, $|m_2|/|m_3| \sim 0.1$, and this will require accidental cancellations of order 10% to take place in the diagonalisation of the lower block of the neutrino mass matrix. Although the degree of tuning necessary to achieve the hierarchy is not large, it does imply that the radiative corrections of the matrix elements will be competetive with the amount by which they need to be tuned, and hence that radiative corrections will be very important in determining the low energy spectrum. In the case of see-saw models [16] the radiative corrections may be sufficient to destroy (or create) the cancellations necessary to achieve the desired hierarchy [17, 18].[1] In all cases radiative corrections will be important and the neutrino masses and mixings calculated in a given high energy unified theory will not be simply related to the low energy ones.

The above situation could be improved if it were possible to make the neutrino mass hierarchy completely natural. If the hierarchy $m_3 \gg m_2$ could emerge without any tuning at all, not even at the level of 10% accidental cancellations, then the low energy spectrum would faithfully preserve the nature of the spectrum calculated at high energy, without being severely affected by radiative corrections. If this could be achieved then the low energy measurements would provide a direct window into the nature of the high energy theory. Therefore it is interesting to ask whether it is possible for the neutrino mass hierarchy to arise in a completely natural way? Indeed this is possible if the see-saw mechanism [16] is supplemented by a mechanism known as single right-handed neutrino dominance [22, 23, 24]. According to single right-handed neutrino dominance one of the right-handed neutrinos makes the dominant contribution to the lower block of m^{ν}_{LL} causing its determinant to approximately vanish, and thereby leading to $|m_2| \ll |m_3|$ without relying on accidental cancellations which are subject to important radiative corrections. Single right-handed neutrino dominance does not mean that there is only a single right-handed neutrino, only that one of the right-handed neutrinos is making the dominant contribution. If the dominant right-handed neutrino is denoted ν_{R3} with a heavy Majorana mass Y and Dirac couplings d, e, f to the weak eigenstate neutrinos ν_e, ν_μ, ν_τ given by $\nu_{R3}(d\nu_e + e\nu_\mu + f\nu_\tau)$ then according to the see-saw mechanism this will result in a light physical neutrino $\nu_3 \approx d\nu_e + e\nu_\mu + f\nu_\tau$ of mass $m_3 \approx (d^2 + e^2 + f^2)/Y$ [22] together with two light orthogonal combinations of neutrinos which would be massless in the limit that there is only a single right-handed neutrino. The requirements of a maximal atmospheric angle $\theta_{23} \approx \pi/4$ and a small CHOOZ angle $\theta_{13} \ll 1$ imply the relation $d \ll e \approx f$ [22].

In order to account for the solar data we must consider the effect of the sub-leading right-handed neutrinos. These perturb the spectrum leading to a small second neutrino mass $m_2^2 \ll m_3^2$. The strength of the hierarchy is controlled by the relative importance of the sub-leading right-handed neutrinos, rather than relying on accidental cancellations. If the sub-leading contribution is dominated by a single sub-leading right-handed neutrino ν_{R2} with mass X and couplings $\nu_{R2}(a\nu_e + b\nu_\mu + c\nu_\tau)$, then this leads to a second neutrino mass of order $m_2 \sim (1/2)(b - c)^2/X$ which only depends on the subleading parameters because we must have $m_2 = 0$ in the limit of there being only a single right-handed

[1]The one-loop beta function coefficients in the Standard Model have recently been corrected in [20, 21]. The two loop result for the MSSM is given in [19].

neutrino [24]. The solar angle was given by $\tan\theta_{12} \sim \sqrt{2}a/(b-c)$ [24]. Note that the solar angle is completely determined by the sub-leading couplings, due to a natural cancellation of the leading contributions [24]. The lightest neutrino mass m_1 is generated by the sub-sub-leading couplings due to the right-handed neutrino ν_{R1} with mass X' leading to a full neutrino mass hierarchy $m_1 \ll m_2 \ll m_3$ [24]. We shall refer to this as sequential sub-dominance.

The sub-leading contribution may alternatively result from two equally contributing sub-dominant right-handed neutrinos in which case only a partial neutrino mass hierarchy results $m_1 \lesssim m_2 \ll m_3$ and the results above will be different [24]. While the full hierarchy results from an approximately diagonal right-handed neutrino mass matrix, a partial hierarchy may result from three possible textures for the right-handed neutrino mass matrix namely diagonal, democratic or off-diagonal where the nomenclature refers to the upper block [23]:

$$M_{RR}^{\rm diag} = \begin{pmatrix} X' & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Y \end{pmatrix} \tag{1.3}$$

$$M_{RR}^{\rm dem} = \begin{pmatrix} X & X & 0 \\ X & X & 0 \\ 0 & 0 & Y \end{pmatrix} \tag{1.4}$$

$$M_{RR}^{\rm off-diag} = \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & Y \end{pmatrix} \tag{1.5}$$

where the heavy Majorana neutrino mass matrices M_{RR} couple right-handed neutrinos to right-handed neutrinos. Note that though the right-handed neutrinos in eq. (1.5) have a common pseudo-Dirac mass X, this may or may not lead to the lighter two physical neutrinos arising from the matrix in eq. (1.1) having a pseudo-Dirac mass. As discussed [24] eq. (1.3) can lead to either a full or a partial hierarchy, while eqs. (1.4) and (1.5) only lead to partial hierarchies. We shall refer to the cases where partial hierarchies result from the structures in eqs. (1.3), (1.4) and (1.5) as diagonal sub-dominance, democratic sub-dominance and off-diagonal sub-dominance. We shall continue to refer the case where a full hierarchy results from the diagonal structure in eq. (1.3) as sequential sub-dominance. Clearly they are all sub-classes of single right-handed neutrino dominance since in all cases the dominant contribution to the neutrino mass matrix comes from the single right-handed neutrino with mass Y. The radiative corrections in such an approach would be expected to be small since the hierarchy is now completely natural, and this has been verified explicitly by a study of each of the cases in eqs. (1.3), (1.4), (1.5) the result of which shows that radiative corrections typically change the physical predictions by only a few per cent [25].

The neutrino mass matrix in eq. (1.2) corresponding to an inverted hierarchy with opposite sign masses $m_1 \approx -m_2$ and $-m_1 \approx m_2 \gg m_3$, can be reproduced by three right-handed neutrinos with the texture in eq. (1.5) [26]. However an important difference is that now it is the off-diagonal pair of right-handed neutrinos with pseudo-Dirac mass X that dominates the neutrino mass matrix leading to the pseudo-Dirac structure of neutrino masses in eq. (1.2), with the right-handed neutrino of mass Y now giving the sub-dominant contributions [26]. We shall refer to this as off-diagonal right-handed neutrino dominance. As in the hierarchical cases based on single right-handed neutrino dominance, the resulting inverted hierarchical spectrum does not rely on any accidental cancellations and is technically natural. The radiative corrections arising to the inverted neutrino mass spectrum arising from off-diagonal right-handed neutrino dominance have been studied and shown to be only a few per cent [26].

In this paper we consider neutrino mass matrices with the leading order structures in eqs. (1.1), (1.2), and diagonalise each of them to leading order in θ_{13} to extract the neutrino masses, mixing angles and phases. The MNS matrix is then constructed to leading order in the small angle θ_{13} including the neutrino *and* charged lepton mixing angles and phases, the latter playing a crucial rôle for allowing the inverted hierarchy solution to be consistent with the LMA MSW solution. We then go on to show how the neutrino mass matrix structures in eqs. (1.1), (1.2) may be constructed naturally from the see-saw mechanism with right-handed neutrino dominance, with no tuning and with small radiative corrections, leading to a full, partial or inverted neutrino mass hierarchy. In each case we derive approximate analytic relations between the input see-saw mass matrices and the resulting neutrino masses, mixing angles and phases. The goal of this analysis is to provide a useful and reliable guide for constructing the LMA MNS matrix in unified models [27].

The analysis builds on that in [24] and [26], by including the effects of phases and the charged lepton mixing angles, which we did not previously consider. It is sufficient in a top-down approach to work to order θ_{13}, since although the radiative corrections are only a few per cent (due to right-handed neutrino dominance), this is sufficient to wash out the order θ_{13}^2 corrections.[2] The leading order results which we present here give the simple relations between the see-saw parameters necessary in order to obtain the phenomenologically successful LMA MNS matrix. Ultimately the masses and mixing angles in unified models must be calculated numerically, including the radiative corrections. The purpose of the analytic results we present here is to provide insight into the construction of unified models which must then be studied numerically.

The paper is organised as follows. In section 2 we review the construction of the MNS matrix starting from general complex neutrino and charged lepton mass matrices, with equivalence of different parametrisations of the MNS matrix discussed in appendix A. We also discuss the charged lepton contributions to the MNS matrix, where the natural expectation is that the charged lepton mixing angles are all small, and give an expansion of the MNS matrix to leading order in the small angles, with details relegated to appendix B. In section 3 we construct the successful neutrino mass matrix structures in a natural way using the see-saw mechanism with right-handed neutrino dominance. In section 3.1 we derive results for single right-handed neutrino dominance with sequential sub-dominance, corresponding to a full neutrino mass hierarchy. In section 3.2 we derive results for single right-handed neutrino dominance with off-diagonal sub-dominance, corresponding to a par-

[2]By contrast in a bottom-up approach the order θ_{13}^2 corrections may be considered [28]. However the effect of charged lepton mixing angles was not considered in [28] and this can significantly affect the conclusions based on naturalness arguments.

tial neutrino mass hierarchy. In section 3.3 we derive results for off-diagonal right-handed neutrino dominance, corresponding to an inverted neutrino mass hierarchy, which we show is consistent with the LMA MSW solution once the effect of charged lepton mixing angles is taken into account. For each type of right-handed neutrino dominance we derive useful analytic expressions for neutrino masses and mixing angles in terms of the see-saw mass matrices, and show that for the hierarchical cases the LMA MSW solution gives a soft lower bound $|U_{e3}| \gtrsim 0.1$, just below the curent CHOOZ limit. The results obtained in section 3 rely on analytic diagonalisation methods discussed in appendices C, D, E. In section 4 we give two physical application of the results. In section 4.1 we consider $\beta\beta_{0\nu}$, and show that both hierchical and inverted hierarchical cases predict small $\beta\beta_{0\nu}$ with $|m_{ee}| \sim 0.007\,\mathrm{eV}$ within the sensitivity of future proposals such as GENIUS. In section 4.2 we discuss leptogenesis, and show that successful leptogenesis is possible if the dominant right-handed neutrino is the heaviest one, but the leptogenesis phase is unrelated to the MNS phases. Section 5 concludes the paper.

2. Constructing the MNS matrix

The charged lepton masses and the neutrino masses are given by the eigenvalues of the complex charged lepton mass matrix m^E_{LR} and the complex symmetric neutrino Majorana matrix m^ν_{LL}, obtained by diagonalising these mass matrices,

$$V^{E_L} m^E_{LR} V^{E_R\dagger} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \tag{2.1}$$

$$V^{\nu_L} m^\nu_{LL} V^{\nu_L T} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \tag{2.2}$$

where V^{E_L}, V^{E_R}, V^{ν_L} are unitary tranformations on the left-handed charged lepton fields E_L, right-handed charged lepton fields E_R, and left-handed neutrino fields ν_L which put the mass matrices into diagonal form with real eigenvalues.

The MNS matrix is then constructed by

$$U_{MNS} = V^{E_L} V^{\nu_L\dagger} \,. \tag{2.3}$$

In appendix A we discuss the relationship between different parametrisations of the MNS matrix. In this paper we find it convenient to parametrise the MNS matrix as

$$U_{MNS} = U_{23} U_{13} U_{12} \tag{2.4}$$

which involves just three irremoveable physical phases δ_{ij}. In this parametrisation the Dirac phase δ which enters the CP odd part of neutrino oscillation probabilities is given by

$$\delta = \delta_{13} - \delta_{23} - \delta_{12} \,. \tag{2.5}$$

where the phases are defined in appendix A.

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The MNS matrix is constructed in eq. (2.3) as a product of a unitary matrix from the charged lepton sector V^{E_L} and a unitary matrix from the neutrino sector $V^{\nu_L\dagger}$. Each of these unitary matrices may be parametrised by the parametrisation of $V^{\dagger}$ in eq. (A.1). Thus we write

$$V^{\nu_L\dagger} = P_2^{\nu_L} R_{23}^{\nu_L} R_{13}^{\nu_L} P_1^{\nu_L} R_{12}^{\nu_L} P_3^{\nu_L} \tag{2.6}$$

$$V^{E_L\dagger} = P_2^{E_L} R_{23}^{E_L} R_{13}^{E_L} P_1^{E_L} R_{12}^{E_L} P_3^{E_L} \tag{2.7}$$

where the Euler angles and phases are defined as in eqs. (A.2)- (A.7) but now there are independent angles and phases for the left-handed neutrino and charged lepton sectors distinguished by the superscripts ν_L and E_L. As shown in appendix B the MNS matrix can be expanded in terms of neutrino and charged lepton mixing angles and phases to leading order in the charged lepton mixing angles which we argue must be small. By comparing eq. (B.12) to eq. (B.11) we have:

$$s_{23}e^{-i\delta_{23}} \approx s_{23}^{\nu_L}e^{-i\delta_{23}^{\nu_L}} - \theta_{23}^{E_L}c_{23}^{\nu_L}e^{-i\delta_{23}^{E_L}} \tag{2.8}$$

$$\theta_{13}e^{-i\delta_{13}} \approx \theta_{13}^{\nu_L}e^{-i\delta_{13}^{\nu_L}} - \theta_{13}^{E_L}c_{23}^{\nu_L}e^{-i\delta_{13}^{E_L}} + \theta_{12}^{E_L}s_{23}^{\nu_L}e^{i(-\delta_{23}^{\nu_L}-\delta_{12}^{E_L})} \tag{2.9}$$

$$s_{12}e^{-i\delta_{12}} \approx s_{12}^{\nu_L}e^{-i\delta_{12}^{\nu_L}} + \theta_{23}^{E_L}s_{12}^{\nu_L}e^{-i\delta_{12}^{\nu_L}} + \theta_{13}^{E_L}c_{12}^{\nu_L}s_{23}^{\nu_L}e^{i(\delta_{23}^{\nu_L}-\delta_{13}^{E_L})} - \theta_{12}^{E_L}c_{23}^{\nu_L}c_{12}^{\nu_L}e^{-i\delta_{12}^{E_L}} \,. \tag{2.10}$$

Clearly the large atmospheric and solar angles θ_{23} and θ_{12} arise mainly from the neutrino sector with small corrections from the charged lepton angles. However θ_{13} receives important contributions not just from $\theta_{13}^{\nu_L}$, but also from the charged lepton angles $\theta_{12}^{E_L}$, and $\theta_{13}^{E_L}$. In models where $\theta_{13}^{\nu_L}$ is extremely small, θ_{13} may originate almost entirely from the charged lepton sector, since for example $\theta_{12}^{E_L}$ may be roughly equal to the Cabibbo angle in some models. Charged lepton contributions could also be important in models where $\theta_{12}^{\nu_L} = \pi/4$ very accurately, since corrections from the charged lepton mixing angles may allow consistency with the LMA MSW solution which requires θ_{12} to be somewhat less than maximal. Such effects will be important for the inverted hierarchy model, for example, as we discuss in more detail later.

3. See-saw models with right-handed neutrino dominance

In this section we specialise to the case of the see-saw mechanism, with right-handed neutrino dominance. A particular high energy theory will involve a charged lepton Yukawa matrix Y^E, a neutrino Yukawa matrix Y^{ν} and a right-handed neutrino Majorana matrix M_{RR}. The Higgs vacuum expectation values (vevs) may be absorbed in the Yukawa matrices to give mass matrices $m_{LR}^E = v_1 Y^E$ and $m_{LR}^{\nu} = v_2 Y^{\nu}$, where we allow for two different vevs v_1, v_2 as is the case in supersymmetric models, while in the standard model $v_1 = v_2$, and LR means that these are Dirac mass matrices in the left-right basis where the rows correspond to left-handed fields and the columns to right-handed fields. At high energies these mass matrices m_{LR}^E, m_{LR}^{ν} and M_{RR} are "born" in a particular basis which is defined by the particular unified theory. In general m_{LR}^E and m_{LR}^{ν} are general complex matrices, while M_{RR} is complex symmetric.

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In this "theory" basis the light effective Majorana neutrino mass matrix is given, up to an overall irrelevant sign, from the see-saw formula [16, 29, 30]:

$$m_{LL}^{\nu} = m_{LR}^{\nu} M_{RR}^{-1} m_{LR}^{\nu\,T} , \tag{3.1}$$

ignoring the effects of radiative corrections.

In order for our results to be maximally useful for top-down model building we must work in the "theory" basis defined by the high energy theory. In this basis all the mass matrices are in general off-diagonal, and in particular the charged lepton mass matrix will be off-diagonal, although according to our naturalness arguments we would expect it to yield small charged lepton mixing angles. Right-handed neutrino dominance should be applied in this theory basis, and questions of naturalness must be addressed within this defining basis. However it may happen that in specific theories certain simplifications naturally appear. For example the right-handed neutrino mass matrix may take one of the approximate forms in eqs. (1.3), (1.4), (1.5).

The general procedure for diagonalising a general complex matrix is discussed in appendix C. The strategy is to diagonalise the charged lepton and neutrino mass matrices m_{LR}^{E} and m_{LL}^{ν} using,

$$P_3^{E_L\,*} R_{12}^{E_L\,T} P_1^{E_L\,*} R_{13}^{E_L\,T} R_{23}^{E_L\,T} P_2^{E_L\,*} m_{LR}^{E} P_2^{E_R} R_{23}^{E_R} R_{13}^{E_R} P_1^{E_R} R_{12}^{E_R} P_3^{E_R} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \tag{3.2}$$

$$P_3^{\nu_L\,*} R_{12}^{\nu_L\,T} P_1^{\nu_L\,*} R_{13}^{\nu_L\,T} R_{23}^{\nu_L\,T} P_2^{\nu_L\,*} m_{LL}^{\nu} P_2^{\nu_L\,*} R_{23}^{\nu_L} R_{13}^{\nu_L} P_1^{\nu_L\,*} R_{12}^{\nu_L} P_3^{\nu_L\,*} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} . \tag{3.3}$$

The specific case of a hierarchical neutrino mass matrix is discussed in appendix D, while the inverted hierarchical case is discussed in appendix E.

3.1 Full hierarchy from sequential sub-dominance

In this sub-section we shall consider the case where the right-handed neutrino mass matrix takes an approximately diagonal form as in eq. (1.3). If it is approximately diagonal, then it may easily be rotated to exactly diagonal form with real eigenvalues, to give

$$M_{RR}^{\rm diag} = \begin{pmatrix} X' & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Y \end{pmatrix} . \tag{3.4}$$

In the theory basis we shall write the complex Dirac mass matrix as

$$m_{LR}^{\nu} = \begin{pmatrix} a' & a & d \\ b' & b & e \\ c' & c & f \end{pmatrix} . \tag{3.5}$$

Then using the see-saw formula (3.1) (valid for complex couplings) we find the complex symmetric neutrino mass matrix,

$$m_{LL}^{\nu} = \begin{pmatrix} \frac{a'^2}{X'} + \frac{a^2}{X} + \frac{d^2}{Y} & \frac{a'b'}{X'} + \frac{ab}{X} + \frac{de}{Y} & \frac{a'c'}{X'} + \frac{ac}{X} + \frac{df}{Y} \\ . & \frac{b'^2}{X'} + \frac{b^2}{X} + \frac{e^2}{Y} & \frac{b'c'}{X'} + \frac{bc}{X} + \frac{ef}{Y} \\ . & . & \frac{c'^2}{X'} + \frac{c^2}{X} + \frac{f^2}{Y} \end{pmatrix} . \tag{3.6}$$

The condition for single right-handed neutrino dominance with sequential sub-dominance was given as [24]

$$\frac{|e^2|,|f^2|,|ef|}{Y} \gg \frac{|xy|}{X} \gg \frac{|x'y'|}{X'}, \tag{3.7}$$

where $x,y \in a,b,c$ and $x',y' \in a',b',c'$, and now we are dealing with complex matrices we must consider the absolute value of the elements. For sequential subdominance we can essentially ignore the contributions from the right-handed neutrino with mass X', so that the problem reduces to the simpler case of two right-handed neutrinos [24] with

$$m^{\nu}_{LL} \approx \begin{pmatrix} \frac{a^2}{X}+\frac{d^2}{Y} & \frac{ab}{X}+\frac{de}{Y} & \frac{ac}{X}+\frac{df}{Y} \\ . & \frac{b^2}{X}+\frac{e^2}{Y} & \frac{bc}{X}+\frac{ef}{Y} \\ . & . & \frac{c^2}{X}+\frac{f^2}{Y} \end{pmatrix} \equiv \begin{pmatrix} m^{\nu}_{11} & m^{\nu}_{12} & m^{\nu}_{13} \\ m^{\nu}_{12} & m^{\nu}_{22} & m^{\nu}_{23} \\ m^{\nu}_{13} & m^{\nu}_{23} & m^{\nu}_{33} \end{pmatrix}, \tag{3.8}$$

where we have made contact with the notation of eq. (D.1). In order that eq. (3.8) resembles the leading order form of the matrix in eq. (1.1) we require

$$|d| \ll |e| \approx |f| \tag{3.9}$$

as first shown in [22], although in the complex case here we have taken the absolute values of the complex couplings. It is straighforward to diagonalise the matrix in eq. (3.8) by following the procedure in appendix D. According to eq. (D.1) there is a complex phase ϕ^{ν}_{ij} associated with each element of the mass matrix. In our previous results the effect of phases was ignored [24]. Here we include them, and discuss their effect on our previous results.

We first focus on the lower block of eq. (3.8),

$$\begin{pmatrix} m^{\nu}_{22} & m^{\nu}_{23} \\ m^{\nu}_{23} & m^{\nu}_{33} \end{pmatrix} \equiv \begin{pmatrix} \frac{b^2}{X}+\frac{e^2}{Y} & \frac{bc}{X}+\frac{ef}{Y} \\ \frac{bc}{X}+\frac{ef}{Y} & \frac{c^2}{X}+\frac{f^2}{Y} \end{pmatrix}. \tag{3.10}$$

This matrix may then be diagonalised by the $P^{\nu_L}_2$ re-phasing followed by the 23 rotation

$$\begin{pmatrix} \tilde{m}^{\nu}_{22} & 0 \\ 0 & m'_3 \end{pmatrix} \equiv R^{\nu_L}_{23}{}^{T} \begin{pmatrix} m^{\nu}_{22}e^{-2i\phi^{\nu_L}_2} & m^{\nu}_{23}e^{-i(\phi^{\nu_L}_2+\phi^{\nu_L}_3)} \\ m^{\nu}_{23}e^{-i(\phi^{\nu_L}_2+\phi^{\nu_L}_3)} & m^{\nu}_{33}e^{-2i\phi^{\nu_L}_3} \end{pmatrix} R^{\nu_L}_{23}. \tag{3.11}$$

The 23 mixing angle may readily be obtained from eq. (D.3), which is accurate to order θ_{13}. We shall find that in the hierarchical cases $\theta_{13} \gtrsim m_2/m_3$ and that this bound must be saturated for the LMA MSW solution. Therefore in this case the results will be accurate to order m_2/m_3. From eq. (D.3) we find

$$\tan 2\theta^{\nu_L}_{23} \approx \frac{2e^{-i(\phi^{\nu}_2+\phi^{\nu}_3)}\frac{ef}{Y}(1+\epsilon_1-\epsilon_2)}{(e^{-2i\phi^{\nu}_3}\frac{f^2}{Y} - e^{-2i\phi^{\nu}_2}\frac{e^2}{Y})} \tag{3.12}$$

where

$$\epsilon_1 = \frac{bc/X}{ef/Y}, \qquad \epsilon_2 = \frac{e^{-2i\phi^{\nu}_3}\frac{c^2}{X} - e^{-2i\phi^{\nu}_2}\frac{b^2}{X}}{e^{-2i\phi^{\nu}_3}\frac{f^2}{Y} - e^{-2i\phi^{\nu}_2}\frac{e^2}{Y}}. \tag{3.13}$$

In the leading approximation, in which the corrections of order ϵ_i are neglected, the phases arise solely from the complex Dirac couplings $e = |e|e^{i\phi_e}$, $f = |f|e^{i\phi_f}$, and we have the elegant leading order result

$$\tan\theta_{23}^{\nu_L} \approx \frac{|e|}{|f|}. \tag{3.14}$$

where the phases are fixed by

$$\phi_2^{\nu_L} - \phi_3^{\nu_L} = \phi_e - \phi_f. \tag{3.15}$$

This leading order result was first written down without phases in [22]. The result in eq. (3.14) demonstrates that the phases on e, f are not important in determining $\theta_{23}^{\nu_L}$ at leading order, only their absolute values matter.

It is instructive to take the determinant of both sides of eq. (3.11),

$$\tilde{m}_{22}^{\nu} m_3' = m_{22}^{\nu} e^{-2i\phi_2^{\nu_L}} m_{33}^{\nu} e^{-2i\phi_3^{\nu_L}} - (m_{23}^{\nu})^2 e^{-2i(\phi_2^{\nu_L}+\phi_3^{\nu_L})} = e^{-2i(\phi_2^{\nu}+\phi_3^{\nu_L})} \frac{(bf-ce)^2}{XY}, \tag{3.16}$$

where the leading order terms proportional to $1/Y^2$ have cancelled in constructing the determinant. This is no accident: it happens because the determinant vanishes in the limit of a single right-handed neutrino. The fact that the determinant is small implies that

$$|m_3'| \gg |\tilde{m}_{22}^{\nu}|. \tag{3.17}$$

The natural origin of eq. (3.17) is crucial both in obtaining a natural hierarchy $m_2 \ll m_3$ and in obtaining a large solar angle. The naturally small 23 subdeterminant is the main consequence of single right-handed neutrino dominance, as emphasised in [23].

If we take the trace of both sides of eq. (3.11), using eq. (3.17), then we may obtain the third neutrino mass to leading order

$$m_3' \approx e^{i(2\phi_e^{\nu}-2\phi_2^{\nu_L})} \frac{[|e|^2+|f|^2]}{Y} \tag{3.18}$$

$$m_3 \approx \frac{[|e|^2+|f|^2]}{Y}. \tag{3.19}$$

A more accurate expression for m_3' including the m_2/m_3 corrections can be readily obtained from eq. (D.8), using eqs. (3.10), (3.12). However the result is not very illuminating, and so is not worth displaying explicitly, although it may be readily constructed if required. The same comments apply to the remaining parameters, which we only display explicitly to leading order.

The leading order result for $\tilde{m}_{22}^{\nu}$, then follows from eqs. (3.18), (3.16), (3.14), (3.15)

$$\tilde{m}_{22}^{\nu} \approx e^{-2i\phi_2^{\nu_L}} \frac{(c_{23}^{\nu_L} b - s_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})})^2}{X}. \tag{3.20}$$

The leading order results for $\tilde{m}_{12}^{\nu}$, $\tilde{m}_{13}^{\nu}$ follow from eq. (D.9), which we write here as

$$\begin{pmatrix} \tilde{m}_{12}^{\nu} \\ \tilde{m}_{13}^{\nu} \end{pmatrix} = R_{23}^{\nu_L T} \begin{pmatrix} m_{12}^{\nu} e^{-i\phi_2^{\nu_L}} \\ m_{13}^{\nu} e^{-i\phi_3^{\nu_L}} \end{pmatrix} \tag{3.21}$$

We find to leading order using eqs. (3.21), (3.8), (3.14), (3.15)

$$\tilde{m}_{12}^{\nu} \approx e^{-i\phi_2^{\nu_L}} \frac{a(c_{23}^{\nu_L} b - s_{23}^{\nu_L} c e^{i(\phi_e^{\nu} - \phi_f^{\nu})})}{X} \tag{3.22}$$

$$\tilde{m}_{13}^{\nu} \approx e^{-i\phi_2^{\nu_L}} \left[\frac{a(s_{23}^{\nu_L} b + c_{23}^{\nu_L} c e^{i(\phi_e^{\nu} - \phi_f^{\nu})})}{X} + e^{i\phi_e} \frac{d\sqrt{|e|^2 + |f|^2}}{Y} \right]. \tag{3.23}$$

Note that the $1/Y$ terms have cancelled to leading order in eq. (3.22) as in the real case [24].

The 13 neutrino angle is given from eq. (D.10)

$$\theta_{13}^{\nu_L} \approx \frac{\tilde{m}_{13}^{\nu}}{m_3'}, \tag{3.24}$$

where the leading order form for $\tilde{m}_{13}^{\nu}$ is given in eq. (3.23), and for m_3' in eq. (3.18). In the large d limit

$$\frac{|de|}{Y}, \frac{|df|}{Y} \gg \frac{|ab|}{X}, \frac{|ac|}{X} \tag{3.25}$$

we find the simple leading order result [24],

$$\theta_{13}^{\nu_L} \approx \frac{|d|}{\sqrt{|e|^2 + |f|^2}} \tag{3.26}$$

with the phase $\phi_2^{\nu_L}$ fixed by

$$\phi_2^{\nu_L} \approx \phi_e - \phi_d . \tag{3.27}$$

Ignoring phases, this was the result previously quoted [24]. On the other hand in the small d limit

$$\frac{|de|}{Y}, \frac{|df|}{Y} \ll \frac{|ab|}{X}, \frac{|ac|}{X} \tag{3.28}$$

we obtain the leading order result

$$\theta_{13}^{\nu_L} \approx \frac{1}{m_3'} e^{-i\phi_2^{\nu_L}} \frac{a(s_{23}^{\nu_L} b + c_{23}^{\nu_L} c e^{i(\phi_e^{\nu} - \phi_f^{\nu})})}{X}, \tag{3.29}$$

where the phase $\phi_2^{\nu_L}$ is fixed by eq. (D.11).

After the 13 rotation the neutrino mass matrix is in block diagonal form

$$\begin{pmatrix} \tilde{m}_{11}^{\nu} & \tilde{m}_{12}^{\nu} & 0 \\ \tilde{m}_{12}^{\nu} & \tilde{m}_{22}^{\nu} & 0 \\ 0 & 0 & m_3' \end{pmatrix}, \tag{3.30}$$

where the leading order form for $\tilde{m}_{12}^{\nu}$ is given in eq. (3.22), for $\tilde{m}_{22}^{\nu}$ in eq. (3.20). We need to find $\tilde{m}_{11}^{\nu}$ from eq. (D.13)

$$\tilde{m}_{11}^{\nu_L} \approx m_{11}^{\nu_L} - \frac{(\tilde{m}_{13}^{\nu_L})^2}{m_3'}. \tag{3.31}$$

Using the leading order form for $\tilde{m}_{13}^{\nu_L}$ in eq. (3.23), and for m_3' in eq. (3.18), and $\tilde{m}_{11}^{\nu_L}$ in eq. (3.8), we find

$$\tilde{m}_{11}^{\nu_L} \approx \frac{a^2}{X} - e^{-i\phi_e^{\nu}} \frac{2d}{\sqrt{|e|^2 + |f|^2}} \cdot \frac{a(s_{23}^{\nu_L} b + c_{23}^{\nu_L} c e^{i(\phi_e^{\nu} - \phi_f^{\nu})})}{X}, \tag{3.32}$$

where the leading $1/Y$ dependence in $\tilde{m}_{11}^{\nu_L}$ has cancelled, as in the real case [24].

In the leading order, valid up to corrections of order m_2/m_3, we may ignore the second term in eq. (3.32) since this cannot be larger than order $\theta_{13}^{\nu_L}$, as shown in eq. (3.26). Then the upper block which remains to be diagonalised in eq. (3.30) is given from eqs. (3.32), (3.22), (3.20), and after the phase transformations $P_1^{\nu_L}$ it is:

$$\begin{pmatrix} A^2 & AB \\ AB & B^2 \end{pmatrix} \tag{3.33}$$

where

$$A = \frac{a}{\sqrt{X}}, \qquad B = e^{-i(\phi_2^{\nu_L}+\chi^{\nu_L})} \frac{(c_{23}^{\nu_L} b - s_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{\sqrt{X}} \tag{3.34}$$

The leading order form in eq. (3.33) clearly has a vanishing determinant

$$m_1' m_2' \approx 0\,. \tag{3.35}$$

This is no surprise since sequential sub-dominance corresponds approximately to the case of two right-handed neutrinos, and hence the lightest neutrino mass is approximately zero, as in the case of a full hierarchy. The situation is analagous to the case of the lower block with single right-handed neutrino dominance discussed above. The second neutrino mass is simply given at leading order from the trace of eq. (3.33),

$$m_2' \approx A^2 + B^2\,. \tag{3.36}$$

The leading order 12 neutrino mixing angle is given from eq. (D.14), (3.33)

$$\tan\theta_{12}^{\nu_L} \approx \frac{A}{B} \approx \frac{a e^{i(\phi_2^{\nu_L}+\chi^{\nu_L})}}{(c_{23}^{\nu_L} b - s_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})})} \tag{3.37}$$

which is analagous to the real result in [24]. The phase χ^{ν_L} is fixed to give a real 12 angle as in eq. (D.15)

$$c_{23}^{\nu_L}|b|\sin(\phi_b') \approx s_{23}^{\nu_L}|c|\sin(\phi_c') \tag{3.38}$$

and then

$$\tan\theta_{12}^{\nu_L} \approx \frac{|a|}{c_{23}^{\nu_L}|b|\cos(\phi_b') - s_{23}^{\nu_L}|c|\cos(\phi_c')}\,, \tag{3.39}$$

where

$$\begin{aligned} \phi_b' &\equiv \phi_b - \phi_a - \phi_2^{\nu} - \chi^{\nu_L}\,, \\ \phi_c' &\equiv \phi_c - \phi_a + \phi_e - \phi_f - \phi_2^{\nu} - \chi^{\nu_L}\,. \end{aligned} \tag{3.40}$$

Eq. (3.39) shows that a large solar angle $\theta_{12}^{\nu_L} \sim \pi/6$ requires $|a|/(|b|\cos(\phi_b')-|c|\cos(\phi_c')) \sim 1$, which is the complex analogue of the real condition $\sqrt{2}a/(b-c) \sim 1$ derived in [24]. The effect of phases is now quite important since the denominator of eq. (3.39) may involve phase-dependent cancellations. In the absence of such cancellations, the basic physical requirement for a large mixing angle, namely that the sub-dominant Dirac coupling a be of order b, c remains as in [24].

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From eqs. (3.18), (3.36), (3.35), we see that sequential sub-dominance has generated a full neutrino mass hierarchy

$$m_1 \ll m_2 \ll m_3 \tag{3.41}$$

and hence

$$\Delta m_{32}^2 \approx m_3^2\,, \qquad \Delta m_{21}^2 \approx m_2^2\,. \tag{3.42}$$

Sequential sub-dominance naturally leads to the LMA MSW solution by assuming a mild hierarchy of couplings in eq. (3.7).

We now discuss an order of magnitude lower bound on the 13 neutrino mixing angle. Clearly such a bound can only be order of magnitude since it is always possible to arrange vanishingly small values of this angle by tuning parameters. However, in the absence of tuning, it is possible to make some general statements about the expected magnitude of this angle. The question is therefore, in the absence of unnatural cancellations, how small can the 13 neutrino angle be? Clearly from eqs. (3.24), (3.23), in the absence of tuning, the smallest values of $\theta_{13}^{\nu_L}$ correspond to setting $d = 0$, and hence the limit will be saturated by eq. (3.29). The 13 neutrino angle in eq. (3.29) cannot be made arbitrarily small by setting $a = 0$ since we have just seen that a large solar angle requires a be of order b, c. Therefore the numerator of eq. (3.29) is expected to be of a similar order of magnitude to m_2 in eq. (3.36), and for a large solar angle we therefore obtain the order of magnitude lower bound

$$|\theta_{13}^{\nu_L}| \gtrsim \left|\frac{m_2}{m_3}\right|. \tag{3.43}$$

From eq. (B.12), since the charged lepton angles are all required to be small due to the naturalness arguments $\theta_{12}^{E_L}, \theta_{13}^{E_L} \lesssim \theta_{13}$, and using eq. (3.42), we may elevate eq. (3.43) to a bound on the MNS element,[3]

$$|U_{e3}|^2 \gtrsim \left|\frac{\Delta m_{21}^2}{\Delta m_{32}^2}\right|. \tag{3.44}$$

The bound is roughly proportional to $\tan\theta_{12}^{\nu_L}$, by eliminating a. Interestingly eq. (3.44) predicts that for the LMA solution $|U_{e3}|$ should be just below the current CHOOZ current limit, which raises the prospect that it could be measured at the forthcoming long baseline (LBL) experiments. We emphasise again, however, that the bound is a soft one since it is based in the premise of there being no cancellations in the construction of the MNS matrix, and so is really only approximate to within an order of magnitude.

3.2 Partial hierarchy from off-diagonal sub-dominance

We now turn the case where at high energies the upper block of the right-handed neutrino mass matrix has the approximate off-diagonal form as in eq. (1.5). By small angle rotations it may then be rotated into the form below with real mass parameters X, Y,

$$M_{RR}^{\mathrm{off-diag}} = \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & Y \end{pmatrix}. \tag{3.45}$$

[3]Similar bounds have been obtained previously in somewhat different frameworks in [31, 33, 28]

Using the same notation for the Dirac couplings as in eq. (3.5), the see-saw formula in eq. (3.1) then gives in this case,

$$m^{\nu}_{LL} = \begin{pmatrix} \frac{2aa'}{X} + \frac{d^2}{Y} & \frac{a'b}{X} + \frac{ab'}{X} + \frac{de}{Y} & \frac{a'c}{X} + \frac{ac'}{X} + \frac{df}{Y} \\ . & \frac{2bb'}{X} + \frac{e^2}{Y} & \frac{b'c}{X} + \frac{bc'}{X} + \frac{ef}{Y} \\ . & . & \frac{2cc'}{X} + \frac{f^2}{Y} \end{pmatrix} \tag{3.46}$$

The condition for single right-handed neutrino dominance with off-diagonal sub-dominance was given as [24]

$$\frac{|e^2|, |f^2|, |ef|}{Y} \gg \frac{|xx'|}{X}\,, \tag{3.47}$$

where $x \in a, b, c$ and $x' \in a', b', c'$, and now we are dealing with complex matrices we must consider the absolute value of the elements.

As before, for the neutrino mass matrix to resemble the leading order form in eq. (1.1) we also require the condition in eq. (3.9). Then the discussion of the leading order 23 neutrino mixing angle follows exactly as in the sequential sub-dominance case, with the lower block replaced by

$$\begin{pmatrix} m^{\nu}_{22} & m^{\nu}_{23} \\ m^{\nu}_{23} & m^{\nu}_{33} \end{pmatrix} \equiv \begin{pmatrix} \frac{2bb'}{X} + \frac{e^2}{Y} & \frac{b'c}{X} + \frac{bc'}{X} + \frac{ef}{Y} \\ \frac{b'c}{X} + \frac{bc'}{X} + \frac{ef}{Y} & \frac{2cc'}{X} + \frac{f^2}{Y} \end{pmatrix}. \tag{3.48}$$

From eq. (D.3) we find the analagous result to eq. (3.12),

$$\tan 2\theta^{\nu_L}_{23} \approx \frac{2e^{-i(\phi^{\nu}_2+\phi^{\nu}_3)}\frac{ef}{Y}(1+\epsilon'_1-\epsilon'_2)}{(e^{-2i\phi^{\nu}_3}\frac{f^2}{Y} - e^{-2i\phi^{\nu}_2}\frac{e^2}{Y})}\,, \tag{3.49}$$

where now

$$\epsilon'_1 = \frac{\frac{b'c}{X} + \frac{bc'}{X}}{\frac{ef}{Y}}\,, \qquad \epsilon'_2 = \frac{e^{-2i\phi^{\nu}_3}\frac{2cc'}{X} - e^{-2i\phi^{\nu}_2}\frac{2bb'}{X}}{e^{-2i\phi^{\nu}_3}\frac{f^2}{Y} - e^{-2i\phi^{\nu}_2}\frac{e^2}{Y}}\,. \tag{3.50}$$

In the leading approximation, in which the corrections of order ϵ_i are neglected, we obtain the same results as in eqs. (3.14), (3.15),

$$\tan\theta^{\nu_L}_{23} \approx \frac{|e|}{|f|}\,. \tag{3.51}$$

The determinant of eq. (3.11) with eq. (3.48) gives

$$\tilde{m}^{\nu}_{22} m'_3 = 2e^{-2i(\phi^{\nu}_2+\phi^{\nu_L}_3)}\frac{(bf-ce)(b'f-c'e)}{XY}\,, \tag{3.52}$$

where again the leading order terms in the determinant have cancelled due to single right-handed neutrino dominance. The trace of eq. (3.11) with eq. (3.48) gives the same leading order result for m'_3 is as in eq. (3.18),

$$m'_3 \approx e^{i(2\phi^{\nu}_e - 2\phi^{\nu_L}_2)}\frac{[|e|^2+|f|^2]}{Y}\,. \tag{3.53}$$

Hence from eqs. (3.52), (3.53),

$$\tilde{m}^{\nu}_{22} \approx 2e^{-2i\phi^{\nu_L}_2}\frac{(c^{\nu_L}_{23}b - s^{\nu_L}_{23}ce^{i(\phi^{\nu}_e-\phi^{\nu}_f)})(c^{\nu_L}_{23}b' - s^{\nu_L}_{23}c'e^{i(\phi^{\nu}_e-\phi^{\nu}_f)})}{X}\,. \tag{3.54}$$

As before, henceforth we display only the leading order results explicitly, with the corrections m_2/m_3 readily obtainable from our formalism if required. The leading order results for $\tilde{m}^{\nu}_{12}$, $\tilde{m}^{\nu}_{13}$ follow from eq. (3.21). We find to leading order using eqs. (3.21), (3.46),

$$\tilde{m}^{\nu}_{12} \approx e^{-i\phi_2^{\nu_L}} \left[\frac{a'(c_{23}^{\nu_L} b - s_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})}) + a(c_{23}^{\nu_L} b' - s_{23}^{\nu_L} c' e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{X} \right] \tag{3.55}$$

$$\tilde{m}^{\nu}_{13} \approx e^{-i\phi_2^{\nu_L}} \left[\frac{a'(s_{23}^{\nu_L} b + c_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{X} + \frac{a(s_{23}^{\nu_L} b' + c_{23}^{\nu_L} c' e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{X} + e^{i\phi_e} \frac{d\sqrt{|e|^2+|f|^2}}{Y} \right]. \tag{3.56}$$

Again the leading order $1/Y$ terms have cancelled in eq. (3.55).

The 13 neutrino angle is given from eq. (D.10) where the leading order form for $\tilde{m}^{\nu}_{13}$ is given in eq. (3.56), and for m_3' in eq. (3.53). In the large d limit we find the same result as in eq. (3.26),

$$\theta_{13}^{\nu_L} \approx \frac{|d|}{\sqrt{|e|^2+|f|^2}}. \tag{3.57}$$

On the other hand in the small d limit we find the leading order result

$$\theta_{13}^{\nu_L} \approx \frac{1}{m_3'} e^{-i\phi_2^{\nu_L}} \left[\frac{a'(s_{23}^{\nu_L} b + c_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{X} + \frac{a(s_{23}^{\nu_L} b' + c_{23}^{\nu_L} c' e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{X} \right], \tag{3.58}$$

where the phase $\phi_2^{\nu_L}$ is fixed by eq. (D.11).

After the 13 rotation the neutrino mass matrix is in block diagonal form as in eq. (3.30). Using the leading order form for $\tilde{m}^{\nu_L}_{13}$ in eq. (3.56), and for m_3' in eq. (3.53), and $\tilde{m}^{\nu_L}_{11}$ in eq. (3.46), we find

$$\tilde{m}^{\nu_L}_{11} \approx \frac{2aa'}{X} - e^{-i\phi_e^{\nu}} \frac{2d}{\sqrt{|e|^2+|f|^2}} \left[\frac{a'(s_{23}^{\nu_L} b + c_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{X} + \frac{a(s_{23}^{\nu_L} b' + c_{23}^{\nu_L} c' e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{X} \right], \tag{3.59}$$

where the leading $1/Y$ dependence in $\tilde{m}^{\nu_L}_{11}$ has cancelled, as before. After the phase transformations $P_1^{\nu_L}$ the upper block which remains to be diagonalised is, approximately,

$$\begin{pmatrix} 2AA' & A'B+AB' \\ A'B+AB' & 2BB' \end{pmatrix} \tag{3.60}$$

where

$$A' = \frac{a'}{\sqrt{X}}, \qquad B' = e^{-i(\phi_2^{\nu_L}+\chi^{\nu_L})} \frac{(c_{23}^{\nu_L} b' - s_{23}^{\nu_L} c' e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}{\sqrt{X}} \tag{3.61}$$

and A, B are as in eq. (3.34). The determinant of eq. (3.60) is now non-zero due to the two sub-dominant right-handed neutrinos,

$$m_1' m_2' \approx -(A'B - AB')^2. \tag{3.62}$$

The trace of eq. (3.60) is,

$$m_1' + m_2' \approx 2(AA' + BB'). \tag{3.63}$$

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The leading order 12 neutrino mixing angle is given from eq. (D.14), (3.60)

$$\tan 2\theta_{12}^{\nu_L} \approx \tan\left(\theta_{12}^{\nu_L \mathrm{seq}} + \theta_{12}^{\nu_L'}\right), \tag{3.64}$$

where the angle $\theta_{12}^{\nu_L \mathrm{seq}}$ is given by

$$\tan\theta_{12}^{\nu_L \mathrm{seq}} \approx \frac{A}{B} \approx \frac{a e^{i(\phi_2^{\nu_L}+\chi^{\nu_L})}}{(c_{23}^{\nu_L} b - s_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})})} \tag{3.65}$$

and $\theta_{12}^{\nu_L'}$ is given by

$$\tan\theta_{12}^{\nu_L'} \approx \frac{A'}{B'} \approx \frac{a' e^{i(\phi_2^{\nu_L}+\chi^{\nu_L})}}{(c_{23}^{\nu_L} b' - s_{23}^{\nu_L} c' e^{i(\phi_e^{\nu}-\phi_f^{\nu})})}\,. \tag{3.66}$$

The phase χ^{ν_L} is fixed by a similar procedure to that described previously.

From eqs. (3.53), (3.62), (3.63), (3.47) we see that off-diagonal sub-dominance has generated a partial neutrino mass hierarchy

$$m_1 \lesssim m_2 \ll m_3 \tag{3.67}$$

and hence

$$\Delta m_{32}^2 \approx m_3^2\,, \qquad \Delta m_{21}^2 = m_2^2 - m_1^2\,. \tag{3.68}$$

Off-diagonal sub-dominance also naturally leads to the LMA MSW solution by assuming a mild hierarchy of couplings in eq. (3.47).

Analagous to the sequential sub-dominance case, there is a lower bound on $\theta_{13}^{\nu_L}$ arising from small d limit limit eq. (3.58). If we assume no accidental cancellations then the square of the numerator in eq. (3.58) is of the same order as the product of eigenvalues in eq. (3.62). Since the lightest two eigenvalues are of similar magnitude in this case, we may deduce a lower bound analagous to eq. (3.44),

$$|U_{e3}|^2 \gtrsim \left|\frac{\Delta m_{21}^2}{\Delta m_{32}^2}\right|. \tag{3.69}$$

3.2.1 The pseudo-Dirac limit

A limiting case of off-diagonal sub-dominance is when the lightest two neutrinos form an approximate pseudo-Dirac pair, which happens when the terms in the upper block in eq. (3.60) satisfy either

$$|A'|, |B| \gg |A|, |B'| \tag{3.70}$$

or

$$|A|, |B'| \gg |A'|, |B|\,. \tag{3.71}$$

The conditions for this happening are, respectively, using eqs. (3.65), (3.66),

$$\tan\theta_{12}^{\nu_L \mathrm{seq}} \ll 1\,, \qquad \tan\theta_{12}^{\nu_L'} \gg 1\,, \tag{3.72}$$

or

$$\tan\theta_{12}^{\nu_L \mathrm{seq}} \gg 1\,, \qquad \tan\theta_{12}^{\nu_L'} \ll 1\,. \tag{3.73}$$

In either of the two limiting cases, from eq. (3.64), we will have an almost maximal neutrino contribution to the solar mixing angle,

$$\theta_{12}^{\nu_L} \approx \frac{\pi}{4} \tag{3.74}$$

and, since the trace of the upper block is very small, opposite sign mass eigenvalues,

$$m_1' \approx -m_2' \tag{3.75}$$

both of which are characteristic features of a pseudo-Dirac neutrino pair. In the pseudo-Dirac case we clearly have

$$\Delta m_{21}^2 \ll m_2^2 \approx m_1^2 \ll m_3^2 \approx \Delta m_{32}^2 \,. \tag{3.76}$$

The hierarchy in eq. (3.76) implies that the pseudo-Dirac case is applicable to the LOW solution rather then the LMA MSW solution, so is not of interest to us here.

Finally, the 13 neutrino mixing angle in the pseudo-Dirac case is given in the small d limit from eq. (3.58), using eqs. (3.70), (3.71),

$$\theta_{13}^{\nu_L} \approx \frac{m_2'}{m_3'} e^{i\chi^{\nu_L}} \,. \tag{3.77}$$

Hence in the pseudo-Dirac case we have the bound analagous to eq. (3.43)

$$|\theta_{13}^{\nu_L}| \gtrsim \left| \frac{m_2}{m_3} \right| . \tag{3.78}$$

This bound differs from that quoted in [28], since it is not singular for $\theta_{12} = \pi/4$.

3.3 Inverted hierarchy from off-diagonal dominance

We now discuss the case of off-diagonal right-handed neutrino dominance which gives rise to a neutrino mass matrix as in eq. (1.2) corresponding to an inverted neutrino mass hierarchy with opposite sign mass eigenvalues for the heavier pseudo-Dirac neutrino pair. The mechanism is based on the off-diagonal form of the right-handed neutrino mass matrix as in eq. (3.45), which after the see-saw mechanism gives rise to the physical neutrino mass matrix in eq. (3.46),

$$m_{LL}^{\nu} = \begin{pmatrix} \frac{2aa'}{X} + \frac{d^2}{Y} & \frac{a'b}{X} + \frac{ab'}{X} + \frac{de}{Y} & \frac{a'c}{X} + \frac{ac'}{X} + \frac{df}{Y} \\ . & \frac{2bb'}{X} + \frac{e^2}{Y} & \frac{b'c}{X} + \frac{bc'}{X} + \frac{ef}{Y} \\ . & . & \frac{2cc'}{X} + \frac{f^2}{Y} \end{pmatrix} . \tag{3.79}$$

However, instead of applying the condition for single right-handed neutrino dominance as in eq. (3.47), we shall now require the pseudo-Dirac pair of right-handed neutrinos with mass X to dominate in such a way as to give the leading order form of the neutrino mass matrix in eq. (1.2). The two alternative conditions for this are either:

$$\frac{|a'b|}{X} \approx \frac{|a'c|}{X} \gg \frac{|a'a|}{X}, \frac{|x'x|}{X}, \frac{|yy'|}{Y} \,, \tag{3.80}$$

where $x' \in b', c'$, and $x \in a, b, c$, and $y, y' \in d, e, f$.

Or:

$$\frac{|ab'|}{X} \approx \frac{|ac'|}{X} \gg \frac{|aa'|}{X}, \frac{|xx'|}{X}, \frac{|yy'|}{Y}, \tag{3.81}$$

where $x \in b, c$, and $x' \in a', b', c'$, and $y, y' \in d, e, f$. In the following we shall consider the first alternative in eq. (3.80) for definiteness.

The procedure for diagonalising the form of the mass matrix in eq. (3.79) with the inverted hierarchy conditions in eq. (3.80) is outlined in appendix (E). We first perform the re-phasing as in eq. (D.2). Then we determine the 23 neutrino mixing angle $\theta_{23}^{\nu_L}$ from eq. (E.5),

$$\tan\theta_{23}^{\nu_L} \approx -e^{i(\phi_2^{\nu_L}-\phi_3^{\nu_L})}\frac{c}{b}(1+\tilde{\epsilon_1}-\tilde{\epsilon_2}), \tag{3.82}$$

where

$$\tilde{\epsilon_1} = \frac{\frac{ac'}{X}+\frac{df}{Y}}{\frac{a'c}{X}}, \qquad \tilde{\epsilon_2} = \frac{\frac{ab'}{X}+\frac{de}{Y}}{\frac{a'b}{X}}. \tag{3.83}$$

The leading order result, neglecting the corrections $\tilde{\epsilon_i}$ is then

$$\tan\theta_{23}^{\nu_L} \approx \frac{|c|}{|b|}, \tag{3.84}$$

where the relative phases are fixed by

$$\phi_2^{\nu_L} - \phi_3^{\nu_L} = \pi + \phi_c^{\nu} - \phi_b^{\nu}. \tag{3.85}$$

We find at leading order

$$\tilde{m}_{12}^{\nu} = e^{i(\phi_b^{\nu}-\phi_2^{\nu_L})}\frac{a'\sqrt{|b|^2+|c|^2}}{X}. \tag{3.86}$$

Henceforth we only give the leading order results explicitly, as in the previous cases. The lower block elements are given by

$$\tilde{m}_{23}^{\nu_L} \approx s_{23}^{\nu_L}c_{23}^{\nu_L}e^{-i2\phi_2^{\nu_L}}\left[\frac{2\sqrt{|b|^2+|c|^2}e^{i\phi_b^{\nu}}(c_{23}^{\nu_L}b' - s_{23}^{\nu_L}c'e^{i(\phi_b^{\nu}-\phi_c^{\nu})})}{X} + \frac{(e^2-f^2e^{2i(\phi_b^{\nu}-\phi_c^{\nu})})}{Y}\right] \tag{3.87}$$

$$\tilde{m}_{22}^{\nu_L} \approx e^{-i2\phi_2^{\nu_L}}\left[\frac{2\sqrt{|b|^2+|c|^2}e^{i\phi_b^{\nu}}(c_{23}^{\nu_L}b' + s_{23}^{\nu_L}c'e^{i(\phi_b^{\nu}-\phi_c^{\nu})})}{X} + \frac{(c_{23}^{\nu_L}e+s_{23}^{\nu_L}fe^{i(\phi_b^{\nu}-\phi_c^{\nu})})^2}{Y}\right] \tag{3.88}$$

$$m_3' \approx e^{-i2\phi_2^{\nu_L}}\frac{(s_{23}^{\nu_L}e - c_{23}^{\nu_L}fe^{i(\phi_b^{\nu}-\phi_c^{\nu})})^2}{Y}. \tag{3.89}$$

We next perform the small angle 13 rotation in eq. (E.9). From eqs. (E.11), (3.86), (3.87) we determine $\theta_{13}^{\nu_L}$,

$$\theta_{13}^{\nu_L} \approx -\frac{\tilde{m}_{23}^{\nu_L}}{\tilde{m}_{12}^{\nu}}. \tag{3.90}$$

The requirement that $\theta_{13}^{\nu_L}$ is real fixes the absolute value of the phases $\phi_2^{\nu_L}$, $\phi_3^{\nu_L}$.

After the $P_1^{\nu_L}$ re-phasing the upper 2×2 block which remains to be diagonalised is of the form

$$\begin{pmatrix}\delta & 1\\ 1 & \epsilon\end{pmatrix} m' \tag{3.91}$$

where from eq. (3.86), (3.88),

$$m' \approx e^{-i\chi^{\nu_L}} \tilde{m}^{\nu}_{12} \tag{3.92}$$

$$\delta \approx \frac{1}{m'}\left(\frac{2aa'}{X} + \frac{d^2}{Y}\right) \tag{3.93}$$

$$\epsilon \approx e^{-i\chi^{\nu_L}} \frac{\tilde{m}^{\nu}_{22}}{\tilde{m}^{\nu}_{12}} . \tag{3.94}$$

Eq. (3.91) is of the pseudo-Dirac form. The trace and determinant of eq. (3.91) reveal the two heavier eigenvalues m'_1, m'_2,

$$m'_1 + m'_2 = (\delta + \epsilon) m' \tag{3.95}$$

$$m'_1 m'_2 = (\delta\epsilon - 1)(m')^2 . \tag{3.96}$$

$$\tag{3.97}$$

Thus we have

$$-m'_1 \approx m'_2 \approx m' \tag{3.98}$$

$$\Delta m^2_{32} \approx -|m'_2|^2 \tag{3.99}$$

$$\Delta m^2_{21} \approx 2|m'_2|^2 \,\mathrm{Re}(\delta + \epsilon) . \tag{3.100}$$

We find the neutrino contribution to the 12 mixing angle by diagonalising eq. (3.91),

$$\tan\theta^{\nu_L}_{12} \approx \frac{2}{\epsilon - \delta} \tag{3.101}$$

which is almost maximal due to the smallness of δ, ϵ or equivalently the pseudo-Dirac form.

From eqs. (3.100), (3.99) we find

$$R \equiv \frac{|\Delta m^2_{21}|}{|\Delta m^2_{32}|} \approx 2\,\mathrm{Re}(\delta + \epsilon) . \tag{3.102}$$

From eq. (3.101) we find

$$1 - \tan\theta^{\nu_L}_{12} \approx \frac{\epsilon - \delta}{2} . \tag{3.103}$$

These relations together with eq. (3.90) give important constraints for the inverted hierarchical case. By comparing eq. (3.94) to eq. (3.90) it is apparent that, in the absence of cancellations, $\theta^{\nu_L}_{13} \sim \epsilon$. The parameter δ is completely independent of $\theta^{\nu_L}_{13}$ and ϵ. Thus for example we can have $\theta^{\nu_L}_{13} = \epsilon = 0$, while at the same time having $\delta \neq 0$ corresponding to $R \neq 0$. Thus, unlike the hierarchical case, there is no lower bound on $\theta^{\nu_L}_{13}$. On the other hand there is an upper bound on $\theta^{\nu_L}_{13}$, in the absence of cancellation effects, which is saturated when δ is not large compared to $\epsilon \sim \theta^{\nu_L}_{13}$. For all choices of $\epsilon \sim \theta^{\nu_L}_{13}$ and δ we find

$$|\theta^{\nu_L}_{13}| \lesssim |1 - \tan\theta^{\nu_L}_{12}| \sim R . \tag{3.104}$$

Since for the LMA MSW solution we know that $R \sim 10^{-2}$, eq. (3.104) implies that the neutrino mixing angles $\theta^{\nu_L}_{12} = \pi/4$ and $\theta^{\nu_L}_{13} = 0$ to high accuracy. However the LMA

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MSW solution is not consistent with maximal mixing and has a preferred mixing angle $\theta_{12} \approx \pi/6$, and for this reason there have been claims in the literature (see for example [28]) that the inverted hierarchy case is inconsistent with the LMA MSW solution. However this conclusion ignores the contribution to MNS angles arising from the charged lepton sector, as was originally pointed out in [26]. It is clear from eqs. (2.9), (2.10) that even if $\theta_{12}^{\nu_L} = \pi/4$ and $\theta_{13}^{\nu_L} = 0$ that a reasonable charged lepton contribution, for example a Cabibbo-like contribution to $\theta_{12}^{E_L} \sim 0.2$, can result in θ_{12} in the LMA MSW range and θ_{13} close to the current CHOOZ limit.

4. Physical applications

4.1 Neutrinoless double beta decay

There has been a recent claim of a signal in neutrinoless double beta decay correponding to $|m_{ee}| = 0.11 - 0.56\,\mathrm{eV}$ at 95% C.L. However this claim has been criticised by two groups [33, 34] and in turn this criticism has been refuted [35]. Since the Heidelberg-Moscow experiment has almost reached its full sensitivity, we may have to wait for a next generation experiment such as GENIUS to resolve this question.

From the theoretical point of view, in the natural hierarchical and inverted hierarchical models favoured here, corresponding to the leading order mass matrices in eqs. (1.1), (1.2), it is immediately clear that very small values of $|m_{ee}| \ll 0.05\,\mathrm{eV}$ are expected, since both matrices have leading order zeroes in the 11 position. Since we argue that these are the only two natural leading order forms, we can immediately predict that the signal for neutrinoless double beta decay should be below the sensitivity of the Heidelberg-Moscow experiment. The next question is whether a more sensitive experiment such as GENIUS is likely to see a signal?

This question has been fully analysed in [33], using the formula

$$|m_{ee}| = \left| \sum_j (U_{ej}^{MNS})^2 m_j \right| \tag{4.1}$$

and the general conclusions are that for the hierarchical models $|m_{ee}| = 0.0005 - 0.005\,\mathrm{eV}$ while for the inverted hierarchical models $|m_{ee}| = 0.01 - 0.057\,\mathrm{eV}$ [33]. It is interesting to re-examine these expectations from the point of view of the equivalent formulation

$$|m_{ee}| = \left| \left(V^{E_L} m_{LL}^{\nu} V^{E_L T} \right)_{11} \right| . \tag{4.2}$$

If we neglect the charged lepton contributions then $|m_{ee}|$ is just given by $|(m_{LL}^{\nu})_{11}|$ and we can read-off the prediction for this in the different models. Of course the contributions from the charged lepton sector can be very important, as we have emphasised, however what can look like a cancellation conspiracy in the formulation in eq. (4.1) can be readily understood in the formulation in eq. (4.2), so the alternative formulation can also be quite instructive, and we use it here as a simple application of our analytic results. There are three case to consider and we discuss each in turn.

We shall show that in all the cases considered, full hierarchy, partial hierarchy, and inverted hierarchy, where the hierarchies result naturally from right-handed neutrino dominance, that neutrinoless double beta decay should have a value of about of $|m_{ee}| \sim 0.007\,\mathrm{eV}$ assuming the LMA MSW solution. Such a value is below the sensitivity of the Heidelberg-Moscow experiment, but is within the range of future proposals such as the GENIUS experiment.

(i) Full hierarchy from sequential subdominance. In this case, ignoring the charged lepton rotations, from eq. (3.8) we find

$$|m_{ee}| \approx |(m^{\nu}_{LL})_{11}| \approx \left|\frac{a^2}{X} + \frac{d^2}{Y}\right| . \tag{4.3}$$

Ignoring the contribution from d^2/Y which is always small even in the large d limit in eq. (3.25), and using eq. (3.36) we find

$$|m_{ee}| \sim m_2 \approx \sqrt{\Delta m^2_{21}} \approx 0.007\,\mathrm{eV} \tag{4.4}$$

which slightly exceeds the upper end of the quoted hierarchical range [33].

(ii) Partial hierarchy from off-diagonal subdominance. Again, ignoring the charged lepton rotations, from eq. (3.46) we have

$$|m_{ee}| \approx |(m^{\nu}_{LL})_{11}| \approx \left|\frac{2aa'}{X} + \frac{d^2}{Y}\right| . \tag{4.5}$$

Again, ignoring the contribution from d^2/Y which is always small, and using eq. (3.63) we again find

$$|m_{ee}| \sim m_2 \approx \sqrt{\Delta m^2_{21}} \approx 0.007\,\mathrm{eV}\,. \tag{4.6}$$

In the pseudo-Dirac limit the term $2aa'/X$ would be very small, so much smaller values of $|m_{ee}|$ would be expected. However, as we discussed earlier, the pseudo-Dirac limit is not relevant for the LMA MSW solution.

(iii) Inverted hierarchy from off-diagonal dominance. Ignoring the charged lepton rotations as before we find, from eq. (3.79),

$$|m_{ee}| \approx |(m^{\nu}_{LL})_{11}| \approx \left|\frac{2aa'}{X} + \frac{d^2}{Y}\right| . \tag{4.7}$$

Using the results of the previous section with eq. (4.7) we find

$$|m_{ee}| \sim |\delta.m_2| \lesssim |Rm_2| \lesssim 0.01 m_2 \lesssim 0.01\sqrt{\Delta m^2_{32}} \lesssim 0.0005\,\mathrm{eV}\,. \tag{4.8}$$

The result in eq. (4.8) neglects the effect of charged lepton mixing angles. However we saw in the previous section that the inverted hierarchy case in inconsistent with the LMA MSW solution unless the effect of charged leptons is considered. Therefore one would expect the result in eq. (4.8) to be considerably affected by charged lepton mixing angles in this case.

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As an example let us consider the effect of the 12 charged lepton mixing angle on the value of $|m_{ee}|$. From eq. (4.2) we find a contribution from $\theta_{12}^{E_L}$,

$$|m_{ee}| \approx \left|\frac{\theta_{12}^{E_L}}{\sqrt{2}}\right| \sqrt{|\Delta m_{32}^2|} \sim \left|\frac{\theta_{12}^{E_L}}{\theta_C}\right| 0.008\,\mathrm{eV}\,, \tag{4.9}$$

where we have scaled the charged lepton angle by the Cabibbo angle θ_C. Thus, if the inverted hierarchy case is to achieve consistency with the LMA MSW solution, we would expect a contribution to neutrinoless double beta decay to be given by eq. (4.9).

4.2 Leptogenesis

Leptogenesis [36, 37] provides another example where a correct treatment of phases is crucial to the physics, and this provides another example of the application of our general treatment including phases considered in this paper. An analytic treatment of leptogenesis in the the framework of single right-handed neutrino dominance was considered earlier [38]. In this subsection we highlight how the more general treatment of phases considered here impacts on our previous analytic estimates. The numerical results previously presented [38] are of course completely unchanged.

According to leptogenesis the baryon asymmetry of the universe is given by

$$Y_B = \frac{\alpha}{\alpha-1} d \frac{\epsilon_1}{g^*}\,, \tag{4.10}$$

where α is a number of order unity arising from sphaleron effects which convert the lepton number to baryon number, d is a dilution factor which is crucial to the success of leptogenesis since it can be many orders of magnitude below unity if the right-handed neutrinos are not produced efficiently enough or if they do not satisfy the decay-out-of-equilibrium condition strongly enough, and $g^* \sim 10^2$ represents the number of effective degrees of freedom. In the analytic discussion here we shall confine our attention to ϵ_1 which is the CP decay asymmetry parameter of the lightest right-handed neutrino, and is the basic seed of lepton number violation in the leptogenesis scenario.

$$\epsilon_1 = \frac{\Gamma(N_{R1} \to L_j + H_2) - \Gamma(N_{R1}^\dagger \to {L_j}^\dagger + H_2^\dagger)}{\Gamma(N_{R1} \to L_j + H_2) + \Gamma(N_{R1}^\dagger \to {L_j}^\dagger + H_2^\dagger)}\,. \tag{4.11}$$

Assuming a hierarchy of right-handed neutrino masses $M_1 \ll M_2 \ll M_3$, one has, approximately,

$$\epsilon_1 \approx -\frac{3}{16\pi((Y_{LR}^\nu)^\dagger Y_{LR}^\nu)_{11}} \sum_{i\neq 1} \mathrm{Im}\left(\left[((Y_{LR}^\nu)^\dagger Y_{LR}^\nu)_{1i}\right]^2\right)\left(\frac{M_1}{M_i}\right). \tag{4.12}$$

Leptogenesis therefore depends on following Yukawa combinations, or, equivalently, Dirac mass combinations

$$((Y_{LR}^\nu)^\dagger Y_{LR}^\nu)_{1i} = \frac{1}{v_2^2}((m_{LR}^\nu)^\dagger m_{LR}^\nu)_{1i}\,. \tag{4.13}$$

The combinations in eq. (4.13) have the property that they are invariant under a change of charged lepton basis, so they may be calculated either in the original theory basis or in the diagonal charged lepton mass basis. However the combinations are sensitive to a basis change of the right-handed neutrinos. For the present discussion we shall assume that in the theory basis the right-handed neutrinos are diagonal as in eq. (1.3). We shall also assume right-handed neutrino dominance with sequential subdominance. Within this class of model, leptogenesis provides a means of discriminanting between different subclasses of sequential subdominance, namely between the case where the dominant right-handed neutrino is the heaviest ($Y \equiv M_3$) and the case where it is the lightest ($Y \equiv M_1$), where we assume $M_1 \ll M_2 \ll M_3$ in both cases. We found that leptogenesis actually prefers the case $Y \equiv M_3$ where the dominant right-handed neutrino is the heaviest [38]. The reason is that leptogenesis is only marginally consistent with the gravitino constraint on the reheat temperature $T_R \leq 10^9$ GeV, and the values of ϵ_1 for the case where the dominant right-handed neutrino is the lighest one are suppressed by a factor of m_2/m_3 relative to the case where it is the heaviest, and this suppression is enough to make it quantitatively unnacceptable [38]. We now consider the two possibilities explicitly.

(a) Sequential subdominance with $Y \ll X \ll X'$. In this case we need to re-order the matrices involving right-handed neutrinos so that the first column corresponds to Y, and so on. Then the combinations in eq. (4.13) can be written as

$$((m_{LR}^{\nu})^{\dagger} m_{LR}^{\nu}) = \begin{pmatrix} \langle z_1|z_1\rangle & \langle z_1|z_2\rangle & \langle z_1|z_3\rangle \\ \langle z_2|z_1\rangle & \langle z_2|z_2\rangle & \langle z_2|z_3\rangle \\ \langle z_3|z_1\rangle & \langle z_3|z_2\rangle & \langle z_3|z_3\rangle \end{pmatrix} \tag{4.14}$$

where we define three complex vectors $|z_1\rangle = (d, e, f)$, $|z_2\rangle = (a, b, c)$, $|z_3\rangle = (a', b', c')$. Clearly the combinations in eq. (4.13) only depend on the first row of the matrix in eq. (4.14), and hence only on two phases, namely that of $\langle z_1|z_2\rangle$ and $\langle z_1|z_3\rangle$.[4] From the sequential subdominance conditions in eqs. (3.7), we find that one term dominates

$$\epsilon_1^{(a)} \approx -\frac{3}{16\pi v_2^2}\frac{Y}{X}\frac{\mathrm{Im}\left(\langle z_1|z_2\rangle^2\right)}{\langle z_1|z_1\rangle} \tag{4.15}$$

so that leptogenesis only depends on one phase combination. Using eq. (3.9), and the leading order results this may be expressed as

$$\epsilon_1^{(a)} \approx -\frac{3}{16\pi v_2^2}\frac{Y}{X}\,\mathrm{Im}\left(e^{-2i\phi_e^{\nu}}\left(s_{23}^{\nu_L} b + c_{23}^{\nu_L} c e^{i(\phi_e^{\nu}-\phi_f^{\nu})}\right)^2\right). \tag{4.16}$$

Using eqs. (3.18), (3.36) we find the order of magnitude result

$$\epsilon_1^{(a)} \sim 2\times 10^{-6}\frac{m_2}{m_3}\sin(2\phi_{12})\left(\frac{Y}{10^{10}\,\mathrm{GeV}}\right), \tag{4.17}$$

where $\phi_{12} = \arg\langle z_1|z_2\rangle$, which quantitatively turns out to be too small due to the m_2/m_3 suppression factor [38].

[4] In general a third phase will enter leptogenesis via $\langle z_2|z_3\rangle$, but in the approximation that lepton number arises from the decay of the lightest right-handed neutrino it is not relevant.

(b) Sequential subdominance with $X' \ll X \ll Y$. The case where the dominant right-handed neutrino is the heaviest requires no re-ordering of the mass matrices, and we have eq. (4.14) with $|z_1\rangle = (a', b', c')$, $|z_2\rangle = (a, b, c)$, $|z_3\rangle = (d, e, f)$ (if $M_1 = X', M_2 = X, M_3 = Y$). From the sequential subdominance conditions in eqs. (3.7), we find the dominant term is now

$$\epsilon_1^{(b)} \approx -\frac{3}{16\pi v_2^2}\frac{X'}{Y}\frac{\mathrm{Im}\left(\langle z_1|z_3\rangle^2\right)}{\langle z_1|z_1\rangle} \tag{4.18}$$

so that leptogenesis now depends on the $\langle z_1|z_3\rangle$ phase combination. Using eq. (3.9), and the leading order results this may be expressed as

$$\epsilon_1^{(b)} \approx -\frac{3}{16\pi v_2^2}\frac{X'}{Y}\frac{|e|^2+|f|^2}{|a'|^2+|b'|^2+|c'|^2}\,\mathrm{Im}\left(e^{2i\phi_e^\nu}(s_{23}^{\nu_L}(b')^* + c_{23}^{\nu_L}(c')^* e^{i(\phi_f^\nu-\phi_e^\nu)})^2\right). \tag{4.19}$$

Using eqs. (3.18), we find the order of magnitude result

$$\epsilon_1^{(b)} \sim 2\times 10^{-6}\sin(2\phi_{13})\left(\frac{X'}{10^{10}\,\mathrm{GeV}}\right) \tag{4.20}$$

where $\phi_{13} = \arg\langle z_1|z_3\rangle$, which quantitatively turns out to be just about acceptable since it does not suffer from the $\frac{m_2}{m_3}$ suppression factor of eq. (4.17) [38].

Note that the leptogenesis phase ϕ_{13} in this case involves the phases of the Dirac couplings to the lightest right-handed neutrino of mass X'. These couplings and phases are completely irrelevant for constructing the MNS matrix.

5. Summary and Conclusion

The latest solar and atmospheric neutrino data strongly point towards a minimal interpretation consisting of three active neutrinos whose mixings are described by a LMA MNS matrix, corresponding to the mixing angles $\theta_{23} \approx \pi/4$, $\theta_{12} \approx \pi/6$, $\theta_{13} \lesssim 0.2$. We have argued on naturalness grounds in favour of a neutrino mass matrix whose leading order form is either eq. (1.1), corresponding to a neutrino mass hierarchy, or eq. (1.2) corresponding to an inverse mass hierarchy with the heavier neutrinos comprising an approximate pseudo-Dirac pair. We have constructed the LMA MNS matrix to leading order in θ_{13} including the neutrino and charged lepton mixing angles and phases, the latter playing a crucial rôle in allowing the inverted hierarchy solution to be consistent with the LMA MSW solution.

We then showed how the neutrino mass matrices with leading order forms in eqs. (1.1) and (1.2) may be constructed naturally from the see-saw mechanism with right-handed neutrino dominance. For the hierarchical form of the mass matrix in eq. (1.1), single right-handed neutrino dominance ensures that we obtain a natural neutrino mass hierarchy $m_2 \ll m_3$ with no tuning, not even at the level of 10% accidental cancellations, which would otherwise be required. Although such a tuning is quite mild, it would mean that the low energy results would be subject to rather large radiative corrections, and therefore that the low energy neutrino masses and mixing angles are not directly related to the high energy ones. By contrast in technically natural theories based on single right-handed neutrino dominance, the radiative corrections are expected to be quite small, at the level of a few per cent [25]. For the inverted hierarchy case, where the leading order form of the mass matrix in eq. (1.2) arises from off-diagonal right-handed neutrino dominance, the radiative corrections are also small [26].

With right-handed neutrino dominance because the radiative corrections are small the low energy observables provide a direct window into the high energy parameters of the theory. In such theories it is therefore meaningful to relate the high energy see-saw parameters to the low energy neutrino observables by simple analytic relations which ignore the radiative corrections. We have done this for three types of right-handed neutrino dominance corresponding to a full hierarchy from single right-handed neutrino dominance with sequential sub-dominance, a partial hierarchy from single right-handed neutrino dominance with off-diagonal subdominance, and an inverted mass hierarchy from off-diagonal dominance. In each case we have derived analytic expressions for neutrino masses, mixing angles and phases in terms of the parameters of the see-saw matrices. Our results are accurate to leading order in θ_{13}, which is sufficiently accurate bearing in mind that such expressions ignore the effects of radiative corrections. The leading order expressions are particularly instructive in guiding the construction of see-saw models, and these should be particularly useful for unified model building. This analysis extends the range of the analytic results in [24, 26] to include the important effects of phases and charged lepton mixing angles which are often overlooked in the literature.

It is worth highlighting how the large atmospheric and solar mixing angles arise for the different types of right-handed neutrino dominance. In all cases we write the neutrino complex Dirac mass matrix in the common notation of eq. (3.5) where the columns of this matrix represent the couplings to the different right-handed neutrinos to the right-handed neutrinos with Majorana mass matrices given in eqs. (1.3), (1.5). Without loss of generality the columns of the matrices may be permuted corresponding to a re-ordering of right-handed neutrinos.

For case of a full neutrino mass hierarchy $m_1 \ll m_2 \ll m_3$ with the right-handed neutrinos having the texture in eq. (1.3) with the right-handed neutrino of mass Y giving the dominant contribution and the right-handed neutrino of mass X giving the sub-dominant contribution, and the right-handed neutrino of mass X' giving negligible contributions, the large atmospheric angle then arises from eq. (3.14), and the large solar angle arises from eq. (3.37). The fact that the solar angle is given, up to phases, by the ratio of the subdominant coupling a to the difference of subdominant couplings $b - c$, was previously known [24]. Eq. (3.39) shows that the cancellation in the denominator in general depends on the relative phases of b, c. In addition the solar angle will receive corrections from the charged lepton sector from eq. (2.10).

For the case of a partial neutrino mass hierarchy $m_1 \lesssim m_2 \ll m_3$ arising from the right-handed neutrinos having the texture in eq. (1.5) with the right-handed neutrino with mass Y dominating and the off-diagonal right-handed neutrinos of pseudo-Dirac mass X giving the sub-dominant contributions, the atmospheric angle is determined similarly to the previous case. However in this case the large solar angle arises from a sum of two angles as in eq. (3.64), where the angle $\theta_{12}^{\nu_L \mathrm{seq}}$ is given by eq. (3.65) while the angle $\theta_{12}^{\nu'_L}$ is given by eq. (3.66). In this case a large solar angle can result from one or both of the angles

$\theta_{12}^{\nu_L \rm seq}$, $\theta_{12}^{\nu_L'}$ being large. As before cancellations can occur in the denominators, which in general depends on the relative phases of b, c and b', c' and in addition the solar angle will receive corrections from the charged lepton sector from eq. (2.10). The pseudo-Dirac limit is not relevant for the LMA MSW solution. For both the full and partial hierarchy cases we obtain a limit $|U_{e3}| \gtrsim 0.1$, just below the current CHOOZ limit.

For the case of an inverted neutrino mass hierarchy with a heavier approximately pseudo-Dirac neutrino pair, $-m_1 \approx m_2 \gg m_3$, arising from the right-handed neutrinos having the texture in eq. (1.5), with now the off-diagonal pseudo-Dirac right-handed neutrinos of mass X dominating, the large atmospheric angle arises from eq. (3.84), assuming the condition in eq. (3.80), similar to [26]. In this case the contribution to the solar angle from the neutrino sector is almost maximal and we have the result in eq. (3.104). However, including the effect of charged lepton mixing angles, for example a Cabibbo-like contribution to $\theta_{12}^{E_L} \sim 0.2$, can result in θ_{12} in the LMA MSW range and θ_{13} close to the current CHOOZ limit, as in the hierarchical case.

Although we only quote the leading order expressions, they should serve to provide a useful guide to unified model building. More accurate expressions including the m_2/m_3 corrections can readily be obtained if required from our more general results.

We have also considered two physical applications of our results, to neutrinoless double beta decay and to leptogenesis. Both hierarchical and inverted hierarchical cases predict small $\beta\beta_{0\nu}$ with $|m_{ee}| \sim 0.007\,\mathrm{eV}$ within the sensitivity of future proposals such as GENIUS. We also considered leptogenesis for the sequential sub-dominance case and emphasised that successful leptogenesis is possible if the dominant right-handed neutrino is the heaviest one, but the leptogenesis phase is unrelated to the MNS phases. Although the leptogenesis phase is not directly related to the MNS phases, it should be emphasised that they all originate from the same source, namely the Dirac mass matrix. It is natural to assume that the neutrino Dirac mass matrix is complex, in which case we would expect both a non-zero leptogenesis phase and non-zero MNS phases. The observation of the Dirac MNS phase at a neutrino factory would therefore be at least circumstantial evidence for a non-zero leptogenesis phase.

To conclude, recent SNO results when combined with other solar and atmosheric neutrino data supports the LMA MSW solution and three active neutrinos. The application of the notion of technical naturalness then leads to two possible mass patterns for neutrinos, either a hierarchy or an inverted hierarchy with an approximately pseudo-Dirac heavier neutrino pair. Further naturalness requirements applied to the see-saw mechanism, leads us to suppose that the successful mass patterns result from right-handed neutrino dominance where one or two right-handed neutrinos dominate and give the leading order mass matrices in eqs. (1.1), (1.2). The maximal atmospheric angle results from a ratio of dominant right-handed neutrino Dirac couplings. The large solar angle results from a ratio of subdominant right-handed neutrino couplings. The leading order results we present here should provide a useful and reliable guide in constructing unified models. The usefulness is due to the simplicity of the leading order results and the reliability is due to the correct treatment of phases and also the fact that the radiative corrections are expected to be relatively small due to the technical naturalness. Both these features are a consequence of

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right-handed neutrino dominance, which may be tested experimentally via its predictions of $\theta_{13} \gtrsim 0.1$ and $|m_{ee}| \sim 0.007$, assuming the LMA MSW solution to be correct, although we emphasise that both predictions are rather soft since they rely on the assumption of the absence of cancellations.

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A. Equivalence of different parametrisations

In this appendix we exhibit the equivalence of different parametrisations of the MNS matrix. A 3×3 unitary matrix may be parametrised by 3 angles and 6 phases. We shall find it convenient to parametrise a unitary matrix $V^\dagger$ by:[5]

$$V^\dagger = P_2 R_{23} R_{13} P_1 R_{12} P_3 \,, \tag{A.1}$$

where R_{ij} are a sequence of real rotations corresponding to the Euler angles θ_{ij}, and P_i are diagonal phase matrices. The Euler matrices are given by

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \tag{A.2}$$

$$R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \tag{A.3}$$

$$R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.4}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. The phase matrices are given by

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.5}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} \tag{A.6}$$

$$P_3 = \begin{pmatrix} e^{i\omega_1} & 0 & 0 \\ 0 & e^{i\omega_2} & 0 \\ 0 & 0 & e^{i\omega_3} \end{pmatrix} . \tag{A.7}$$

[5] It is convenient to define the parametrisation of $V^\dagger$ rather than V because the MNS matrix involves ${V^{\nu_L}}^\dagger$ and the neutrino mixing angles will play a central rôle.

By commuting the phase matrices to the left, it is not difficult to show that the parametrisation in eq. (A.1) is equivalent to

$$V^\dagger = P U_{23} U_{13} U_{12}\,, \tag{A.8}$$

where $P = P_1 P_2 P_3$ and

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{-i\delta_{23}} \\ 0 & -s_{23}e^{i\delta_{23}} & c_{23} \end{pmatrix} \tag{A.9}$$

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \tag{A.10}$$

$$U_{12} = \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_{12}} & 0 \\ -s_{12}e^{i\delta_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.11}$$

where

$$\delta_{23} = \chi + \omega_2 - \omega_3 \tag{A.12}$$

$$\delta_{13} = \omega_1 - \omega_3 \tag{A.13}$$

$$\delta_{12} = \omega_1 - \omega_2 \tag{A.14}$$

The matrix U_{MNS} is an example of a unitary matrix, and as such it may be parametrised by either of the equivalent forms in eqs. (A.1) or (A.8). If we use the form in eq. (A.8) then the phase matrix P on the left may always be removed by an additional charged lepton phase rotation $\Delta V^{E_L} = P^\dagger$, which is always possible since right-handed charged lepton phase rotations can always make the charged lepton masses real. Therefore U_{MNS} can always be parametrised by

$$U_{MNS} = U_{23} U_{13} U_{12} \tag{A.15}$$

which involves just three irremoveable physical phases δ_{ij}. In this parametrisation the Dirac phase δ which enters the CP odd part of neutrino oscillation probabilities is given by

$$\delta = \delta_{13} - \delta_{23} - \delta_{12}\,. \tag{A.16}$$

Another common parametrisation of the MNS matrix is

$$U_{MNS} = R_{23} U_{13} R_{12} P_0\,, \tag{A.17}$$

where

$$P_0 = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.18}$$

and in eq. (A.17) U_{13} is of the form in eq. (A.10) but with δ_{13} replaced by the Dirac phase δ. The parametrisation in eq. (A.17) can be transformed into the parametrisation in eq. (A.15) by commuting the phase matrix P_0 in eq. (A.17) to the left, and then removing the phases on the left-hand side by charged lepton phase rotations. The two parametrisations are then related by the phase relations

$$\delta_{23} = \beta_2 \tag{A.19}$$
$$\delta_{13} = \delta + \beta_1 \tag{A.20}$$
$$\delta_{12} = \beta_1 - \beta_2\,. \tag{A.21}$$

The use of the parametrisation in eq. (A.17) is widespread in the literature, however for the reasons discussed in the next sub-section we prefer to use the parametrisation in eq. (A.15) which is trivially related to eq. (A.17) by the above phase relations.

B. Charged lepton contributions to the MNS matrix

In this appendix we discuss the contribution of the charged lepton mixing angles to the MNS matrix. The MNS matrix is constructed in eq. (2.3) as a product of a unitary matrix from the charged lepton sector V^{E_L} and a unitary matrix from the neutrino sector $V^{\nu_L\dagger}$. Each of these unitary matrices may be parametrised by the parametrisation of $V^\dagger$ in eq. (A.1). Thus we write

$$V^{\nu_L\dagger} = P_2^{\nu_L} R_{23}^{\nu_L} R_{13}^{\nu_L} P_1^{\nu_L} R_{12}^{\nu_L} P_3^{\nu_L} \tag{B.1}$$
$$V^{E_L\dagger} = P_2^{E_L} R_{23}^{E_L} R_{13}^{E_L} P_1^{E_L} R_{12}^{E_L} P_3^{E_L}\,, \tag{B.2}$$

where the Euler angles and phases are defined as in eqs. (A.2)–(A.7) but now there are independent angles and phases for the left-handed neutrino and charged lepton sectors distinguished by the superscripts ν_L and E_L. The MNS matrix from eqs. (2.3), (B.1), (B.2) is then

$$U_{MNS} = P_3^{E_L\dagger} R_{12}^{E_L\dagger} P_1^{E_L\dagger} R_{13}^{E_L\dagger} R_{23}^{E_L\dagger} P_2^{E_L\dagger} P_2^{\nu_L} R_{23}^{\nu_L} R_{13}^{\nu_L} P_1^{\nu_L} R_{12}^{\nu_L} P_3^{\nu_L}\,. \tag{B.3}$$

As before we commute all the phase matrices to the left, then choose $P_3^{E_L\dagger}$ to cancel all the phases on the left-hand side, to leave just

$$U_{MNS} = U_{12}^{E_L\dagger} U_{13}^{E_L\dagger} U_{23}^{E_L\dagger} U_{23}^{\nu_L} U_{13}^{\nu_L} U_{12}^{\nu_L} \tag{B.4}$$

with independent phases and angles for the left-handed neutrino and charged lepton sectors, in the convention of eqs. (A.9), (A.10), (A.11). The phases in eq. (B.4) are given in terms of the phases in eqs. (B.1), (B.2) by

$$\delta_{12}^{\nu_L} = \omega_1^{\nu_L} - \omega_2^{\nu_L} \tag{B.5}$$
$$\delta_{13}^{\nu_L} = \omega_1^{\nu_L} - \omega_3^{\nu_L} \tag{B.6}$$
$$\delta_{23}^{\nu_L} = \chi^{\nu_L} + \omega_2^{\nu_L} - \omega_3^{\nu_L} \tag{B.7}$$
$$\delta_{23}^{E_L} = -\phi_2^{E_L} + \phi_3^{E_L} + \phi_2^{\nu_L} - \phi_3^{\nu_L} + \chi^{\nu_L} + \omega_2^{\nu_L} - \omega_3^{\nu_L} \tag{B.8}$$
$$\delta_{13}^{E_L} = \phi_3^{E_L} - \phi_3^{\nu_L} + \omega_1^{\nu_L} - \omega_3^{\nu_L} \tag{B.9}$$
$$\delta_{12}^{E_L} = \chi^{E_L} + \phi_2^{E_L} - \phi_2^{\nu_L} - \chi^{\nu_L} + \omega_1^{\nu_L} - \omega_2^{\nu_L}\,. \tag{B.10}$$

The form of U_{MNS} in eq. (B.4) is similar to the parametrisation in eq. (2.4), which is the practical reason why we prefer that form rather than that in eq. (A.17).

We now discuss the MNS matrix to leading order in θ_{13}. From eqs. (A.15), (A.9), (A.10) and (A.11), we find to leading order in θ_{13} that U_{MNS} may be expanded as:

$$
\begin{aligned}
U_{MNS} &\approx \\
&\approx \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_{12}} & \theta_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23}e^{i\delta_{12}} - c_{12}s_{23}\theta_{13}e^{i(\delta_{13}-\delta_{23})} & c_{12}c_{23} - s_{12}s_{23}\theta_{13}e^{i(-\delta_{23}+\delta_{13}-\delta_{12})} & s_{23}e^{-i\delta_{23}} \\ s_{12}s_{23}e^{i(\delta_{23}+\delta_{12})} - c_{12}c_{23}\theta_{13}e^{i\delta_{13}} & -c_{12}s_{23}e^{i\delta_{23}} - s_{12}c_{23}\theta_{13}e^{i(\delta_{13}-\delta_{12})} & c_{23} \end{pmatrix}.
\end{aligned}
\tag{B.11}
$$

For $\theta_{13} = 0.1$, close to the CHOOZ limit, the approximate form in eq. (B.11) is accurate to 1%.

We now wish to expand the MNS matrix in terms of neutrino and charged lepton mixing angles and phases to leading order in small angles, using eq. (B.4). In technically natural theories, based on right-handed neutrino dominance, the contribution to θ_{23} comes mainly from the neutrino sector, $\theta_{23} \approx \theta_{23}^{\nu_L}$. Furthermore in natural theories we expect that the contributions to θ_{13} are all separately small so that the smallness of this angle does not rely on accidental cancellations. Clearly this implies that $\theta_{13}^{\nu_L}$ and $\theta_{13}^{E_L}$ must both be $\lesssim \theta_{13}$. Since the 13 element of U_{MNS} also receives a contribution from the charged lepton sector proportional to $s_{12}^{E_L}s_{23}^{\nu_L}$, the same argument also implies that $\theta_{12}^{E_L} \lesssim \theta_{13}$. Therefore the natural expectation is that all the charged lepton mixing angles are small! Expanding eq. (B.4) to leading order in small angles $\theta_{12}^{E_L}$, $\theta_{23}^{E_L}$, $\theta_{13}^{E_L}$, $\theta_{13}^{\nu_L}$, we find

$$
\begin{aligned}
U_{MNS} &\approx \\
&\approx \begin{pmatrix} c_{12}^{\nu_L} & s_{12}^{\nu_L}e^{-i\delta_{12}^{\nu_L}} & \theta_{13}^{\nu_L}e^{-i\delta_{13}^{\nu_L}} \\ -s_{12}^{\nu_L}c_{23}^{\nu_L}e^{i\delta_{12}^{\nu_L}} - c_{12}^{\nu_L}s_{23}^{\nu_L}\theta_{13}^{\nu_L}e^{i(\delta_{13}^{\nu_L}-\delta_{23}^{\nu_L})} & c_{12}^{\nu_L}c_{23}^{\nu_L} - s_{12}^{\nu_L}s_{23}^{\nu_L}\theta_{13}^{\nu_L}e^{i(-\delta_{23}^{\nu_L}+\delta_{13}^{\nu_L}-\delta_{12}^{\nu_L})} & s_{23}^{\nu_L}e^{-i\delta_{23}^{\nu_L}} \\ s_{12}^{\nu_L}s_{23}^{\nu_L}e^{i(\delta_{23}^{\nu_L}+\delta_{12}^{\nu_L})} - c_{12}^{\nu_L}c_{23}^{\nu_L}\theta_{13}^{\nu_L}e^{i\delta_{13}^{\nu_L}} & -c_{12}^{\nu_L}s_{23}^{\nu_L}e^{i\delta_{23}^{\nu_L}} - s_{12}^{\nu_L}c_{23}^{\nu_L}\theta_{13}^{\nu_L}e^{i(\delta_{13}^{\nu_L}-\delta_{12}^{\nu_L})} & c_{23}^{\nu_L} \end{pmatrix} + \\
&+ \theta_{23}^{E_L}\begin{pmatrix} c_{12}^{\nu_L} & s_{12}^{\nu_L}e^{-i\delta_{12}^{\nu_L}} & 0 \\ -s_{23}^{\nu_L}s_{12}^{\nu_L}e^{i(\delta_{23}^{\nu_L}-\delta_{23}^{E_L}+\delta_{12}^{\nu_L})} & s_{23}^{\nu_L}c_{12}^{\nu_L}e^{i(\delta_{23}^{\nu_L}-\delta_{23}^{E_L})} & -c_{23}^{\nu_L}e^{-i\delta_{23}^{E_L}} \\ -c_{23}^{\nu_L}s_{12}^{\nu_L}e^{i(\delta_{23}^{E_L}+\delta_{12}^{\nu_L})} & c_{23}^{\nu_L}c_{12}^{\nu_L}e^{i\delta_{23}^{E_L}} & s_{23}^{\nu_L}e^{i(\delta_{23}^{E_L}-\delta_{23}^{\nu_L})} \end{pmatrix} + \\
&+ \theta_{13}^{E_L}\begin{pmatrix} -s_{12}^{\nu_L}s_{23}^{\nu_L}e^{i(\delta_{12}^{\nu_L}+\delta_{23}^{\nu_L}-\delta_{13}^{E_L})} & c_{12}^{\nu_L}s_{23}^{\nu_L}e^{i(\delta_{23}^{\nu_L}-\delta_{13}^{E_L})} & -c_{23}^{\nu_L}e^{-i\delta_{13}^{E_L}} \\ 0 & 0 & 0 \\ c_{12}^{\nu_L}e^{i\delta_{13}^{E_L}} & s_{12}^{\nu_L}e^{i(-\delta_{12}^{\nu_L}+\delta_{13}^{E_L})} & 0 \end{pmatrix} + \\
&+ \theta_{12}^{E_L}\begin{pmatrix} c_{23}^{\nu_L}s_{12}^{\nu_L}e^{i(\delta_{12}^{\nu_L}-\delta_{12}^{E_L})} & -c_{23}^{\nu_L}c_{12}^{\nu_L}e^{-i\delta_{12}^{E_L}} & s_{23}^{\nu_L}e^{i(-\delta_{23}^{\nu_L}-\delta_{12}^{E_L})} \\ c_{12}^{\nu_L}e^{i\delta_{12}^{E_L}} & s_{12}^{\nu_L}e^{i(-\delta_{12}^{\nu_L}+\delta_{12}^{E_L})} & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{B.12}
$$

where we have dropped terms of order $\theta_{23}^{E_L}\theta_{13}$. The first matrix on the right hand side of eq. (B.12) gives the contribution to the MNS matrix from the neutrino mixing angles and phases, and is of the same form as eq. (B.11). The subsequent matrices give the corrections to the MNS matrix from the charged lepton mixing angles $\theta_{23}^{E_L}$, $\theta_{13}^{E_L}$, and $\theta_{12}^{E_L}$.

C. Procedure for diagonalising hierarchical mass matrices

In this appendix we discuss the diagonalisation of a general complex hierarchical matrix m where

$$m = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}. \tag{C.1}$$

The matrix m is diagonalised by a sequence of tranformations:

$$P_3^{L*} {R_{12}^L}^T P_1^{L*} {R_{13}^L}^T {R_{23}^L}^T P_2^{L*} m P_2^R R_{23}^R R_{13}^R P_1^R R_{12}^R P_3^R = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{C.2}$$

In the case of the charged lepton mass matrix, all the rotation angles are small, while in the case of the neutrino mass matrix it is symmetric. The results for the general complex matrix m will be sufficiently general to allow us to apply them to both of the physical cases of interest as limiting cases.

The procedure for diagonalising a general hierarchical matrix m involves the following steps.

1. The first step involves multiplying the mass matrix m by the inner phase matrices P_2 defined in eq. (A.6):

$$P_2^{L*} m P_2^R = \begin{pmatrix} m_{11} & m_{12}e^{i\phi_2^R} & m_{13}e^{i\phi_3^R} \\ m_{21}e^{-i\phi_2^L} & m_{22}e^{i(\phi_2^R-\phi_2^L)} & m_{23}e^{i(\phi_3^R-\phi_2^L)} \\ m_{31}e^{-i\phi_3^L} & m_{32}e^{i(\phi_2^R-\phi_3^L)} & m_{33}e^{i(\phi_3^R-\phi_3^L)} \end{pmatrix} \equiv \begin{pmatrix} m_{11} & m'_{12} & m'_{13} \\ m'_{21} & m'_{22} & m'_{23} \\ m'_{31} & m'_{32} & m'_{33} \end{pmatrix}. \tag{C.3}$$

The purpose of this re-phasing is to facilitate steps 2,3 using real rotation angles θ_{23}, θ_{13}, as we shall see.

2. The second step is to perform the real rotations R_{23} defined in eq. (A.2) on the re-phased matrix from step1. The purpose is to put zeroes in the 23,32 elements of the resulting matrix:

$${R_{23}^L}^T \begin{pmatrix} m_{11} & m'_{12} & m'_{13} \\ m'_{21} & m'_{22} & m'_{23} \\ m'_{31} & m'_{32} & m'_{33} \end{pmatrix} R_{23}^R \equiv \begin{pmatrix} m_{11} & \tilde{m}_{12} & \tilde{m}_{13} \\ \tilde{m}_{21} & \tilde{m}_{22} & 0 \\ \tilde{m}_{31} & 0 & m'_3 \end{pmatrix}. \tag{C.4}$$

The zeroes in the 23,32 positions are achieved by diagonalising the lower 23 block, using the reduced matrix R_{23} obtained by striking out the row and column in which the unit element appears, to leave a 2×2 rotation,

$${R_{23}^L}^T \begin{pmatrix} m'_{22} & m'_{23} \\ m'_{32} & m'_{33} \end{pmatrix} R_{23}^R \equiv \begin{pmatrix} \tilde{m}_{22} & 0 \\ 0 & m'_3 \end{pmatrix} \tag{C.5}$$

which implies

$$\tan 2\theta_{23}^L = \frac{2\left[m'_{33}m'_{23} + m'_{22}m'_{32}\right]}{\left[{m'_{33}}^2 - {m'_{22}}^2 + {m'_{32}}^2 - {m'_{23}}^2\right]} \tag{C.6}$$

$$\tan 2\theta_{23}^R = \frac{2\left[m'_{33}m'_{32} + m'_{22}m'_{23}\right]}{\left[{m'_{33}}^2 - {m'_{22}}^2 + {m'_{23}}^2 - {m'_{32}}^2\right]}. \tag{C.7}$$

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The requirement that the angles θ_{23}^L and θ_{23}^R are real means that the numerators and denominators must have equal phases, and this is achieved by adjusting the relative phases $\phi_i^R - \phi_j^L$ which appear in the lower block of eq. (C.3). The remaining elements are then given by the reduced rotations

$$\begin{pmatrix} \tilde{m}_{12} & \tilde{m}_{13} \end{pmatrix} = \begin{pmatrix} m'_{12} & m'_{13} \end{pmatrix} R_{23}^R \tag{C.8}$$

$$\begin{pmatrix} \tilde{m}_{21} \\ \tilde{m}_{31} \end{pmatrix} = R_{23}^{L\ T} \begin{pmatrix} m'_{21} \\ m'_{31} \end{pmatrix}. \tag{C.9}$$

3. The third step is to perform the real small angle rotations R_{13} defined in eq. (A.3) on the matrix from step 2. The purpose is to put zeroes in the 13,31 elements of the resulting matrix:

$$R_{13}^{L\ T} \begin{pmatrix} m_{11} & \tilde{m}_{12} & \tilde{m}_{13} \\ \tilde{m}_{21} & \tilde{m}_{22} & 0 \\ \tilde{m}_{31} & 0 & m'_3 \end{pmatrix} R_{13}^R \approx \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} & 0 \\ \tilde{m}_{21} & \tilde{m}_{22} & 0 \\ 0 & 0 & m'_3 \end{pmatrix} \tag{C.10}$$

The zeroes in the 13,31 positions are achieved by diagonalising the outer 13 block, using the reduced matrix R_{13} obtained by striking out the row and column in which the unit element appears, to leave a 2×2 rotation,

$$R_{13}^{L\ T} \begin{pmatrix} m_{11} & \tilde{m}_{13} \\ \tilde{m}_{31} & m'_3 \end{pmatrix} R_{13}^R \approx \begin{pmatrix} \tilde{m}_{11} & 0 \\ 0 & m'_3 \end{pmatrix} \tag{C.11}$$

which implies

$$\theta_{13}^L \approx \frac{\tilde{m}_{13}}{m'_3} + \frac{\tilde{m}_{31} m_{11}}{(m'_3)^2} \tag{C.12}$$

$$\theta_{13}^R \approx \frac{\tilde{m}_{31}}{m'_3} + \frac{\tilde{m}_{13} m_{11}}{(m'_3)^2}. \tag{C.13}$$

The requirement that the angles θ_{13}^L and θ_{31}^R are real fixes the absolute value of the phases $\phi_i^R + \phi_j^L$, since only the relative phases were fixed previously. This uses up all the phase freedom and thus all the resulting mass matrix elements in eq. (C.10) remain complex. Note that eq. (C.10) is written to leading order in the small angles θ_{13}, and as discussed previously the 23,32 elements remain zero to this order. The large complex element m'_3 is approximately unchanged to this order. Due to the zeroes in the 23,32 position of the matrix the elements $\tilde{m}_{12}$ and $\tilde{m}_{21}$ are also unchanged to leading order. The element $\tilde{m}_{22}$ is also unchanged of course since it is not present in the reduced matrix. The only new element is therefore

$$\tilde{m}_{11} \approx m_{11} - \frac{\tilde{m}_{13}\tilde{m}_{31}}{m'_3}. \tag{C.14}$$

4. The fourth step involves multiplying the mass matrix resulting from eq. (C.10) by the phase matrices P_1 defined in eq. (A.5):

$$\begin{aligned} P_1^{L*} \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} & 0 \\ \tilde{m}_{21} & \tilde{m}_{22} & 0 \\ 0 & 0 & m_3' \end{pmatrix} P_1^R &= \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12}e^{i\chi^R} & 0 \\ \tilde{m}_{21}e^{-i\chi^L} & \tilde{m}_{22}e^{i(\chi^R-\chi^L)} & 0 \\ 0 & 0 & m_3' \end{pmatrix} \\ &\equiv \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12}' & 0 \\ \tilde{m}_{21}' & \tilde{m}_{22}' & 0 \\ 0 & 0 & m_3' \end{pmatrix}. \end{aligned} \tag{C.15}$$

The purpose of this re-phasing is to facilitate step 5 using real rotation angle θ_{12}.

5. The fifth step is to perform the real rotations R_{12} defined in eq. (A.4) on the re-phased matrix from step 4. The purpose is to put zeroes in the 12,21 elements of the resulting matrix:

$$R_{12}^{L\,T} \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12}' & 0 \\ \tilde{m}_{21}' & \tilde{m}_{22}' & 0 \\ 0 & 0 & m_3' \end{pmatrix} R_{12}^R \equiv \begin{pmatrix} m_1' & 0 & 0 \\ 0 & m_2' & 0 \\ 0 & 0 & m_3' \end{pmatrix}. \tag{C.16}$$

The zeroes in the 12,21 positions are achieved by diagonalising the upper 12 block, using the reduced matrix R_{12} obtained by striking out the row and column in which the unit element appears, to leave a 2×2 rotation,

$$R_{12}^{L\,T} \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12}' \\ \tilde{m}_{21}' & \tilde{m}_{22}' \end{pmatrix} R_{12}^R \equiv \begin{pmatrix} m_1' & 0 \\ 0 & m_2' \end{pmatrix} \tag{C.17}$$

which implies

$$\tan 2\theta_{12}^L = \frac{2\left[\tilde{m}_{22}'\tilde{m}_{12}' + \tilde{m}_{11}\tilde{m}_{21}'\right]}{\left[(\tilde{m}_{22}')^2 - (\tilde{m}_{11})^2 + (\tilde{m}_{21}')^2 - (\tilde{m}_{12}')^2\right]} \tag{C.18}$$

$$\tan 2\theta_{12}^R = \frac{2\left[\tilde{m}_{22}'\tilde{m}_{21}' + \tilde{m}_{11}\tilde{m}_{12}'\right]}{\left[(\tilde{m}_{22}')^2 - (\tilde{m}_{11})^2 + (\tilde{m}_{12}')^2 - (\tilde{m}_{21}')^2\right]}. \tag{C.19}$$

The requirement that the angles θ_{12}^L and θ_{21}^R are real means that the numerators and denominators must have equal phases, and this is achieved by adjusting the phases χ_L, χ_R.

6. The sixth step involves multiplying the complex diagonal mass matrix resulting from eq. (C.16) by the phase matrices P_3 defined in eq. (A.7):

$$P_3^{L*} \begin{pmatrix} m_1' & 0 & 0 \\ 0 & m_2' & 0 \\ 0 & 0 & m_3' \end{pmatrix} P_3^R = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{C.20}$$

The result of this re-phasing is a diagonal matrix with real eigenvalues. In the case of charged leptons this last step can be achieved by a suitable P_3^R for any choice of P_3^L. This freedom in P_3^L enables three phases to be removed from the MNS matrix.

D. Diagonalising the hierarchical neutrino mass matrix

In this appendix we shall apply the results of appendix C to the case of the complex symmetric hierarchical neutrino mass matrix of the leading order form as shown in eq. (1.1), which will be written in full generality as

$$m^{\nu}_{LL} = \begin{pmatrix} m^{\nu}_{11} & m^{\nu}_{12} & m^{\nu}_{13} \\ m^{\nu}_{12} & m^{\nu}_{22} & m^{\nu}_{23} \\ m^{\nu}_{13} & m^{\nu}_{23} & m^{\nu}_{33} \end{pmatrix} \equiv \begin{pmatrix} |m^{\nu}_{11}|e^{i\phi^{\nu}_{11}} & |m^{\nu}_{12}|e^{i\phi^{\nu}_{12}} & |m^{\nu}_{13}|e^{i\phi^{\nu}_{13}} \\ |m^{\nu}_{12}|e^{i\phi^{\nu}_{12}} & |m^{\nu}_{22}|e^{i\phi^{\nu}_{22}} & |m^{\nu}_{23}|e^{i\phi^{\nu}_{23}} \\ |m^{\nu}_{13}|e^{i\phi^{\nu}_{13}} & |m^{\nu}_{23}|e^{i\phi^{\nu}_{23}} & |m^{\nu}_{33}|e^{i\phi^{\nu}_{33}} \end{pmatrix}, \tag{D.1}$$

where it should be remembered that the elements in the lower 23 block are larger than the other elements, as in eq. (1.1).

The procedure outlined in appendix D for diagonalising m^{ν}_{LL} is to work our way from the inner transformations to the outer transformations as follows.

1. Re-phase m^{ν}_{LL} using the $P^{\nu_L}_2$.
2. Put zeroes in the $23 = 32$ positions using $R^{\nu_L}_{23}$.
3. Put zeroes in the $13 = 31$ positions using $R^{\nu_L}_{13}$.
4. Re-phase the mass matrix using $P^{\nu_L}_1$.
5. Put zeroes in the $12 = 21$ positions using $R^{\nu_L}_{12}$.
6. Make the diagonal elements real using the $P^{\nu_L}_3$.

If $\theta^{\nu_L}_{13}$ is small, then for the hierarchical case $m_3 \gg m_2$ this procedure will result in an approximately diagonal matrix to leading order in $\theta^{\nu_L}_{13}$. One might object that after step 3 the $R^{\nu_L}_{13}$ rotations will "fill-in" the zeroes in the 23,32 positions with terms of order $\theta^{\nu_L}_{13}$ multiplied by $m^{\nu_L}_{12}, m^{\nu_L}_{13}$. However in the hierarchical case $m^{\nu_L}_{12}, m^{\nu_L}_{13}$ are smaller than $m^{\nu_L}_{33}$ by a factor of $\theta^{\nu_L}_{13}$ which means that the "filled-in" 23,32 entries are suppressed by a total factor of $(\theta^{\nu_L}_{13})^2$ compared to the 33 element. This means that after the 5 steps above a hierarchical matrix will be diagonal to leading order in $\theta^{\nu_L}_{13}$, as claimed. For the inverted hierarchical neutrino case a different procedure must be followed, as discussed in the next sub-section. Here we shall systematically diagonalise the hierarchical neutrino mass matrix in eq. (D.1) by following the above procedure as follows.

The first step is to re-phase the matrix in eq. (D.1) using $P^{\nu_L *}_2$ so that the neutrino mass matrix becomes,

$$\begin{pmatrix} |m^{\nu}_{11}|e^{i\phi^{\nu}_{11}} & |m^{\nu}_{12}|e^{i(\phi^{\nu}_{12}-\phi^{\nu_L}_2)} & |m^{\nu}_{13}|e^{i(\phi^{\nu}_{13}-\phi^{\nu_L}_3)} \\ |m^{\nu}_{12}|e^{i(\phi^{\nu}_{12}-\phi^{\nu_L}_2)} & |m^{\nu}_{22}|e^{i(\phi^{\nu}_{22}-2\phi^{\nu_L}_2)} & |m^{\nu}_{23}|e^{i(\phi^{\nu}_{23}-\phi^{\nu_L}_2-\phi^{\nu_L}_3)} \\ |m^{\nu}_{13}|e^{i(\phi^{\nu}_{13}-\phi^{\nu_L}_3)} & |m^{\nu}_{23}|e^{i(\phi^{\nu}_{23}-\phi^{\nu_L}_2-\phi^{\nu_L}_3)} & |m^{\nu}_{33}|e^{i(\phi^{\nu}_{33}-2\phi^{\nu_L}_3)} \end{pmatrix}. \tag{D.2}$$

To determine the 23 neutrino mixing angle $\theta^{\nu_L}_{23}$ we perform a 23 rotation which diagonalises the lower 23 block of eq. (D.2). From eq. (C.6) we find the 23 neutrino mixing angle $\theta^{\nu_L}_{23}$ as

$$\tan 2\theta^{\nu_L}_{23} = \frac{2\left[|m^{\nu}_{23}|e^{i(\phi^{\nu}_{23}-\phi^{\nu_L}_2-\phi^{\nu_L}_3)}\right]}{\left[|m^{\nu}_{33}|e^{i(\phi^{\nu}_{33}-2\phi^{\nu_L}_3)} - |m^{\nu}_{22}|e^{i(\phi^{\nu}_{22}-2\phi^{\nu_L}_2)}\right]}. \tag{D.3}$$

The relative phase $\phi_2^{\nu_L} - \phi_3^{\nu_L}$ is fixed by the requirement that the angle $\theta_{23}^{\nu_L}$ in eq. (D.3) be real,

$$|m_{33}^{\nu}|\sin(\phi_{33}^{\nu} - \phi_{23}^{\nu} + \phi_2^{\nu_L} - \phi_3^{\nu_L}) = |m_{22}^{\nu}|\sin(\phi_{22}^{\nu} - \phi_{23}^{\nu} + \phi_3^{\nu_L} - \phi_2^{\nu_L})\,. \tag{D.4}$$

After the 23 rotation in eq. (C.4), the neutrino mass matrix in eq. (D.2) becomes

$$\begin{pmatrix} m_{11}^{\nu} & \tilde{m}_{12}^{\nu} & \tilde{m}_{13}^{\nu} \\ \tilde{m}_{12}^{\nu} & \tilde{m}_{22}^{\nu} & 0 \\ \tilde{m}_{13}^{\nu} & 0 & m_3' \end{pmatrix} \tag{D.5}$$

The lower block elements are given by

$$\begin{pmatrix} \tilde{m}_{22}^{\nu} & 0 \\ 0 & m_3' \end{pmatrix} \equiv R_{23}^{\nu_L T} \begin{pmatrix} |m_{22}^{\nu}|e^{i(\phi_{22}^{\nu} - 2\phi_2^{\nu_L})} & |m_{23}^{\nu}|e^{i(\phi_{23}^{\nu} - \phi_2^{\nu_L} - \phi_3^{\nu_L})} \\ |m_{23}^{\nu}|e^{i(\phi_{23}^{\nu} - \phi_2^{\nu_L} - \phi_3^{\nu_L})} & |m_{33}^{\nu}|e^{i(\phi_{33}^{\nu} - 2\phi_3^{\nu_L})} \end{pmatrix} R_{23}^{\nu_L} \tag{D.6}$$

which implies

$$\begin{aligned} \tilde{m}_{22}^{\nu} &= (c_{23}^{\nu_L})^2|m_{22}^{\nu}|e^{i(\phi_{22}^{\nu} - 2\phi_2^{\nu_L})} - 2s_{23}^{\nu_L}c_{23}^{\nu_L}|m_{23}^{\nu}|e^{i(\phi_{23}^{\nu} - \phi_2^{\nu_L} - \phi_3^{\nu_L})} + \\ &\quad + (s_{23}^{\nu_L})^2|m_{33}^{\nu}|e^{i(\phi_{33}^{\nu} - 2\phi_3^{\nu_L})} \end{aligned} \tag{D.7}$$

$$\begin{aligned} m_3' &= (s_{23}^{\nu_L})^2|m_{22}^{\nu}|e^{i(\phi_{22}^{\nu} - 2\phi_2^{\nu_L})} + 2s_{23}^{\nu_L}c_{23}^{\nu_L}|m_{23}^{\nu}|e^{i(\phi_{23}^{\nu} - \phi_2^{\nu_L} - \phi_3^{\nu_L})} + \\ &\quad + (c_{23}^{\nu_L})^2|m_{33}^{\nu}|e^{i(\phi_{33}^{\nu} - 2\phi_3^{\nu_L})} \end{aligned} \tag{D.8}$$

and from eq. (C.9)

$$\begin{pmatrix} \tilde{m}_{12}^{\nu} \\ \tilde{m}_{13}^{\nu} \end{pmatrix} = R_{23}^{\nu_L T} \begin{pmatrix} |m_{12}^{\nu}|e^{i(\phi_{12}^{\nu} - \phi_2^{\nu_L})} \\ |m_{13}^{\nu}|e^{i(\phi_{13}^{\nu} - \phi_3^{\nu_L})} \end{pmatrix}\,. \tag{D.9}$$

We now perform a 13 rotation on the neutrino matrix in eq. (D.5) which diagonalises the outer 13 block of eq. (D.5) and determines the 13 neutrino mixing angle $\theta_{13}^{\nu_L}$. From eq. (C.12) we find the 13 neutrino mixing angle $\theta_{13}^{\nu_L}$ as

$$\theta_{13}^{\nu_L} \approx \frac{\tilde{m}_{13}^{\nu}}{m_3'}\,. \tag{D.10}$$

The absolute phases $\phi_2^{\nu_L}$, $\phi_3^{\nu_L}$ are fixed by the requirement that the angle $\theta_{13}^{\nu_L}$ in eq. (D.10) be real,

$$s_{23}^{\nu_L}|m_{12}^{\nu}|\sin(\phi_{12}^{\nu} - \phi_2^{\nu_L} - \phi_3') + c_{23}^{\nu_L}|m_{13}^{\nu}|\sin(\phi_{13}^{\nu} - \phi_3^{\nu_L} - \phi_3') = 0\,. \tag{D.11}$$

After the 13 rotation in eq. (C.10), eq. (D.5) becomes

$$\begin{pmatrix} \tilde{m}_{11}^{\nu} & \tilde{m}_{12}^{\nu} & 0 \\ \tilde{m}_{12}^{\nu} & \tilde{m}_{22}^{\nu} & 0 \\ 0 & 0 & m_3' \end{pmatrix} \equiv \begin{pmatrix} |\tilde{m}_{11}^{\nu}|e^{i\tilde{\phi}_{11}^{\nu}} & |\tilde{m}_{12}^{\nu}|e^{i\tilde{\phi}_{12}^{\nu}} & 0 \\ |\tilde{m}_{12}^{\nu}|e^{i\tilde{\phi}_{12}^{\nu}} & |\tilde{m}_{22}^{\nu}|e^{i\tilde{\phi}_{22}^{\nu}} & 0 \\ 0 & 0 & |m_3'|e^{i\phi_3'} \end{pmatrix}\,. \tag{D.12}$$

To leading order in $\theta_{13}^{\nu_L}$ the only new element in eq. (D.12) is

$$\tilde{m}_{11}^{\nu_L} \approx m_{11}^{\nu_L} - \frac{(\tilde{m}_{13}^{\nu_L})^2}{m_3'}\,. \tag{D.13}$$

It only remains to determine the 12 neutrino mixing angle $\theta_{12}^{\nu_L}$ by diagonalising the upper 12 block of eq. (D.12). From eq. (C.18) we find the 12 neutrino mixing angle $\theta_{12}^{\nu_L}$ as

$$\tan 2\theta_{12}^{\nu_L} = \frac{2\left[|\tilde{m}_{12}^{\nu}|e^{i(\tilde{\phi}_{12}^{\nu}-\chi^{\nu_L})}\right]}{\left[|\tilde{m}_{22}^{\nu}|e^{i(\tilde{\phi}_{22}^{\nu}-2\chi^{\nu_L})} - |\tilde{m}_{11}^{\nu}|e^{i\tilde{\phi}_{11}^{\nu}}\right]}\,. \tag{D.14}$$

The phase χ^{ν_L} is fixed by the requirement that the angle $\theta_{12}^{\nu_L}$ in eq. (D.14) be real,

$$|\tilde{m}_{22}^{\nu}|\sin(\tilde{\phi}_{22}^{\nu} - \tilde{\phi}_{12}^{\nu} - \chi^{\nu_L}) = |\tilde{m}_{11}^{\nu}|\sin(\tilde{\phi}_{11}^{\nu} - \tilde{\phi}_{12}^{\nu} + \chi^{\nu_L})\,. \tag{D.15}$$

After the 12 rotation the upper block of the matrix in eq. (D.12) is diagonal and the resulting matrix is

$$\begin{pmatrix} m_1' & 0 & 0 \\ 0 & m_2' & 0 \\ 0 & 0 & m_3' \end{pmatrix} \equiv \begin{pmatrix} m_1 e^{i\phi_1'} & 0 & 0 \\ 0 & m_2 e^{i\phi_2'} & 0 \\ 0 & 0 & m_3 e^{i\phi_3'} \end{pmatrix} \tag{D.16}$$

where from eq. (C.17)

$$m_1' = (c_{12}^{\nu_L})^2|\tilde{m}_{11}^{\nu}|e^{i\tilde{\phi}_{11}^{\nu}} - 2s_{12}^{\nu_L}c_{12}^{\nu_L}|\tilde{m}_{12}^{\nu}|e^{i(\tilde{\phi}_{12}^{\nu}-\chi^{\nu_L})} + (s_{12}^{\nu_L})^2|\tilde{m}_{22}^{\nu}|e^{i(\tilde{\phi}_{22}^{\nu}-2\chi^{\nu_L})} \tag{D.17}$$

$$m_2' = (s_{12}^{\nu_L})^2|\tilde{m}_{11}^{\nu}|e^{i\tilde{\phi}_{11}^{\nu}} + 2s_{12}^{\nu_L}c_{12}^{\nu_L}|\tilde{m}_{12}^{\nu}|e^{i(\tilde{\phi}_{12}^{\nu}-\chi^{\nu_L})} + (c_{12}^{\nu_L})^2|\tilde{m}_{22}^{\nu}|e^{i(\tilde{\phi}_{22}^{\nu}-2\chi^{\nu_L})}\,. \tag{D.18}$$

It is a simple matter to adjust the phases $\omega_i^{\nu_L}$ in $P_3^{\nu_L}$ to remove the phases in eq. (D.16) and make the neutrino masses real, as in eq. (C.20),

$$\omega_i^{\nu_L} = \frac{\phi_i'}{2}\,. \tag{D.19}$$

This completes the diagonalisation in eq. (3.3). In the case of neutrino masses, unlike the case of the charged fermions, there is no left over phase freedom. This is the reason why the MNS matrix has three more physical phases than the CKM matrix.

E. Diagonalising the inverted hierarchical neutrino mass matrix

In this appendix we shall consider the case of the complex symmetric inverted hierarchical neutrino mass matrix of the leading order form as shown in eq. (1.2). In this case the procedure is as follows.

1. Re-phase m_{LL}^{ν} using the $P_2^{\nu_L}$.
2. Put zeroes in the $13 = 31$ positions using $R_{23}^{\nu_L}$.
3. Put zeroes in the $23 = 32$ positions using $R_{13}^{\nu_L}$.
4. Re-phase the mass matrix using $P_1^{\nu_L}$.
5. Put zeroes in the $12 = 21$ positions using $R_{12}^{\nu_L}$.
6. Make the diagonal elements real using the $P_3^{\nu_L}$.

We continue to write the neutrino mass matrix as in eq. (D.1), but now it should be remembered that the 12,13 elements are now larger than the other elements, as in eq. (1.2). This is the reason why the above procedure differs from that for the case of the hierarchical neutrino mass matrix.

We first perform the re-phasing as in eq. (D.2). Then we determine the 23 neutrino mixing angle $\theta_{23}^{\nu_L}$ by performing a 23 rotation such that

$$\begin{pmatrix} m_{11}^{\nu} & \tilde{m}_{12}^{\nu} & 0 \\ \tilde{m}_{12}^{\nu} & \tilde{m}_{22}^{\nu} & \tilde{m}_{23}^{\nu} \\ 0 & \tilde{m}_{23}^{\nu} & m_3' \end{pmatrix} \equiv$$
$$\equiv R_{23}^{\nu_L T} \begin{pmatrix} |m_{11}^{\nu}|e^{i\phi_{11}^{\nu}} & |m_{12}^{\nu}|e^{i(\phi_{12}^{\nu}-\phi_2^{\nu_L})} & |m_{13}^{\nu}|e^{i(\phi_{13}^{\nu}-\phi_3^{\nu_L})} \\ |m_{12}^{\nu}|e^{i(\phi_{12}^{\nu}-\phi_2^{\nu_L})} & |m_{22}^{\nu}|e^{i(\phi_{22}^{\nu}-2\phi_2^{\nu_L})} & |m_{23}^{\nu}|e^{i(\phi_{23}^{\nu}-\phi_2^{\nu_L}-\phi_3^{\nu_L})} \\ |m_{13}^{\nu}|e^{i(\phi_{13}^{\nu}-\phi_3^{\nu_L})} & |m_{23}^{\nu}|e^{i(\phi_{23}^{\nu}-\phi_2^{\nu_L}-\phi_3^{\nu_L})} & |m_{33}^{\nu}|e^{i(\phi_{33}^{\nu}-2\phi_3^{\nu_L})} \end{pmatrix} R_{23}^{\nu_L}\,, \tag{E.1}$$

where

$$\begin{pmatrix} \tilde{m}_{12}^{\nu} \\ 0 \end{pmatrix} = R_{23}^{\nu_L T} \begin{pmatrix} |m_{12}^{\nu}|e^{i(\phi_{12}^{\nu}-\phi_2^{\nu_L})} \\ |m_{13}^{\nu}|e^{i(\phi_{13}^{\nu}-\phi_3^{\nu_L})} \end{pmatrix} \tag{E.2}$$

which gives the 23 neutrino mixing angle $\theta_{23}^{\nu_L}$ in this case to be

$$\tan\theta_{23}^{\nu_L} = \frac{-|m_{13}^{\nu}|e^{i(\phi_{13}^{\nu}-\phi_3^{\nu_L})}}{|m_{12}^{\nu}|e^{i(\phi_{12}^{\nu}-\phi_2^{\nu_L})}}\,. \tag{E.3}$$

Since the Euler angles are constrained to satisfy $\theta_{ij} \leq \pi/2$, we must have $\tan\theta_{23}^{\nu_L} \approx +1$, and this then fixes

$$\phi_{13}^{\nu} - \phi_{12}^{\nu} + \phi_2^{\nu_L} - \phi_3^{\nu_L} = \pi \tag{E.4}$$

This fixes $\phi_2^{\nu_L} - \phi_3^{\nu_L}$ and gives

$$\tan\theta_{23}^{\nu_L} = \frac{|m_{13}^{\nu}|}{|m_{12}^{\nu}|} \tag{E.5}$$

and

$$\tilde{m}_{12}^{\nu} = c_{23}^{\nu_L}|m_{12}^{\nu}|e^{i(\phi_{12}^{\nu}-\phi_2^{\nu_L})} - s_{23}^{\nu_L}|m_{13}^{\nu}|e^{i(\phi_{13}^{\nu}-\phi_3^{\nu_L})}\,. \tag{E.6}$$

The lower block elements are given by

$$\begin{pmatrix} \tilde{m}_{22}^{\nu} & \tilde{m}_{23}^{\nu} \\ \tilde{m}_{23}^{\nu} & m_3' \end{pmatrix} \equiv R_{23}^{\nu_L T} \begin{pmatrix} |m_{22}^{\nu}|e^{i(\phi_{22}^{\nu}-2\phi_2^{\nu_L})} & |m_{23}^{\nu}|e^{i(\phi_{23}^{\nu}-\phi_2^{\nu_L}-\phi_3^{\nu_L})} \\ |m_{23}^{\nu}|e^{i(\phi_{23}^{\nu}-\phi_2^{\nu_L}-\phi_3^{\nu_L})} & |m_{33}^{\nu}|e^{i(\phi_{33}^{\nu}-2\phi_3^{\nu_L})} \end{pmatrix} R_{23}^{\nu_L} \tag{E.7}$$

which implies

$$\begin{aligned} \tilde{m}_{23}^{\nu} &= s_{23}^{\nu_L}c_{23}^{\nu_L}\left(|m_{22}^{\nu}|e^{i(\phi_{22}^{\nu}-2\phi_2^{\nu_L})} - |m_{33}^{\nu}|e^{i(\phi_{33}^{\nu}-2\phi_3^{\nu_L})}\right) + \\ &\quad + \left((c_{23}^{\nu_L})^2 - (s_{23}^{\nu_L})^2\right)|m_{23}^{\nu}|e^{i(\phi_{23}^{\nu}-\phi_2^{\nu_L}-\phi_3^{\nu_L})} \end{aligned} \tag{E.8}$$

and the remaining diagonal elements are given as before in eqs. (D.7), (D.8).

We next perform a small angle 13 rotation such that

$$\begin{pmatrix} m_{11}^{\nu} & \tilde{m}_{12}^{\nu} & 0 \\ \tilde{m}_{12}^{\nu} & \tilde{m}_{22}^{\nu} & 0 \\ 0 & 0 & m_3' \end{pmatrix} \approx R_{13}^{\nu_L T} \begin{pmatrix} m_{11}^{\nu} & \tilde{m}_{12}^{\nu} & 0 \\ \tilde{m}_{12}^{\nu} & \tilde{m}_{22}^{\nu} & \tilde{m}_{23}^{\nu} \\ 0 & \tilde{m}_{23}^{\nu} & m_3' \end{pmatrix} R_{13}^{\nu_L}\,, \tag{E.9}$$

where

$$\begin{pmatrix} \tilde{m}^{\nu}_{12} \\ 0 \end{pmatrix} \approx R^{\nu_L\,T}_{13} \begin{pmatrix} \tilde{m}^{\nu}_{12} \\ \tilde{m}^{\nu}_{23} \end{pmatrix} . \tag{E.10}$$

Note that to leading order in $\theta^{\nu_L}_{13}$ the large element $\tilde{m}^{\nu}_{12}$ is unchanged. The remaining elements in eq. (E.9) are also unchanged to leading order in $\theta^{\nu_L}_{13}$. The $13 = 31$ element in eq. (E.9) gets filled in by a term $\theta^{\nu_L}_{13}(m^{\nu}_{11} - m'_3)$ which is of order $(\theta^{\nu_L}_{13})^2$ compared to $\tilde{m}^{\nu}_{12}$ and does not appear to leading order in $\theta^{\nu_L}_{13}$. From eq. (E.10) the 13 neutrino mixing angle $\theta^{\nu_L}_{13}$ is

$$\theta^{\nu_L}_{13} \approx \frac{-\tilde{m}^{\nu}_{23}}{\tilde{m}^{\nu}_{12}} . \tag{E.11}$$

The requirement that $\theta^{\nu_L}_{13}$ is real fixes the absolute value of the phases $\phi^{\nu_L}_2$, $\phi^{\nu_L}_3$.

The left hand side of eq. (E.9) now resembles the left hand side of eq. (D.12), except that here m^{ν}_{11} is unchanged due to the zero $13 = 31$ element after the 23 rotation. Therefore the rest of the diagonalisation process follows that of the previous hierarchical case from eq. (D.14) onwards, where now

$$\tan 2\theta^{\nu_L}_{12} = \frac{2\left[|\tilde{m}^{\nu}_{12}|e^{i(\tilde{\phi}^{\nu}_{12} - \chi^{\nu_L})}\right]}{\left[|\tilde{m}^{\nu}_{22}|e^{i(\tilde{\phi}^{\nu}_{22} - 2\chi^{\nu_L})} - |m^{\nu}_{11}|e^{i\phi^{\nu}_{11}}\right]} . \tag{E.12}$$

Note that in the inverted hierarchy case here we have

$$|\tilde{m}^{\nu}_{12}| \gg |\tilde{m}^{\nu}_{22}|, |m^{\nu}_{11}| \tag{E.13}$$

which implies an almost degenerate pair of pseudo-Dirac neutrino masses (with opposite sign eigenvalues) and an almost maximal 12 mixing angle from eq. (E.12).

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